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COLLEGE ALGEBRA.

BY

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THE UNIVERSITY OF CHICAGO.

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PREFACE.

THIS text is intended primarily for the college and the technical school. By treating only the subjects usually given in the college course in algebra, space has been gained for a more detailed exposition of the more difficult topics. As the extent and the character of the review at the beginning of the course upon topics prescribed for entrance to college varies so widely, and as the review is usually conducted in an informal manner, it seemed best to the author to leave to the instructor the review of the elementary principles, but to give in the text review exercises.

As to the order of the subjects, the aim has been to present first those topics which are readily mastered by the student, and to reserve for the latter part of the text the questions grouped around the subjects involving infinite series. In reviewing the subject of simultaneous equations, the student is led naturally, almost unavoidably, to the determinant notation. Determinants are therefore treated early in the text, an order of presentation shown by actual experience to give very satisfactory results, especially in arousing the interest of the student.

In the chapter on graphic algebra, the first principles of coördinate geometry are introduced and applied to the study of simultaneous equations and inequalities. In this connection is presented an elementary account of the solution of numerical equations, chiefly from the graphic standpoint. The arrangement is such as to admit of a very brief course or of a fuller course involving Horner's method of synthetic division. The practice of emphasizing graphic algebra in courses for technical students commends itself also to the general student.

Attention is invited to the proofs of the fundamental theorems on logarithms, the treatment of mathematical induction and the illustrations showing the necessity of the different steps in the process, the examples from the physical sciences of the topic variation, the complete proof of the general binomial theorem independent of the principle "permanence of form," the establishment of the relations between the roots and the coefficients of the quadratic, cubic, and quartic equations prior to the proof of the general theorem, the solution of those equations before introducing the assumption that every equation has a root (here proved in the Appendix).

The attempt has been made to present the subjects limits and infinite series in as simple form as is consistent with rigor.

Forty-five sets of exercises, averaging over fifteen to a set, are given at very short intervals in the text. Some historical data have been introduced, with no attempt to give the source, the subject-matter being classical.

The author is under great obligation to Dr. Moulton, of the Department of Astronomy of the University of Chicago, who read with care the entire manuscript and offered numerous suggestions as to the form of presentation, most of which have been adopted. Likewise the thanks of the author are due Professor J. W. A. Young of the same University, who examined the more critical chapters and offered valuable suggestions.

Proof-sheets of the entire book were carefully read by Professor L. L. Conant of the Worcester Polytechnic Institute, whose corrections and suggestions have led to numerous improvements. Finally, the author is indebted to Professors Moore and Young of the University of Chicago, who examined critically the proof-sheets of certain portions of the text.

CHICAGO, January, 1902.

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SYMBOLS AND ABBREVIATIONS.

$=$, equal.

$\neq$, not equal.

$\doteq$, approaches, 106.

∞, infinity, 69, 105.

$e = 2.71828\ldots$, 137.

$i = \sqrt{-1}$.

${}_nC_r$, ${}_nP_r$, 86.

A.P., G.P., H.P., 64.

sin, cos, 208.

$|t| =$ absolute value of t, 114.

$n! = 1 \times 2 \times 3 \times \ldots \times n$, 86.

COLLEGE ALGEBRA.

CHAPTER I.

NUMBER IN ALGEBRA; SURDS AND IMAGINARIES.

1. **The natural numbers** or **positive integers** (1, 2, 3, ...) make it possible to enumerate the objects of a group considered for the time as equivalent entities. It has been established that primitive counting was done on the fingers and that in many languages the numeral 5 is merely the word for hand, 10 for both hands, and 20 for the whole man (hands and feet).* While 5, 10, 20, and 60 have been used as bases, 10 is the usual base.

If a group of objects can be partitioned into equivalent smaller groups, each smaller group or a combination of them is a **fraction** ($\frac{1}{2}$, $\frac{2}{3}$, $\frac{4}{5}$, ...) of the original group. Abstractly, a fraction is the quotient of two positive integers. Fractional results may or may not admit of an interpretation in a particular problem. A shepherd would declare it to be impossible to separate his flock of 50 sheep into three equal flocks; but would find no theoretical difficulty in dividing a 50-foot rope into three equal pieces. The algebraic statement for each problem is the same: to find x such that

---

* Thus in the language of the Tamanacs, the word for 6 is "one on the other hand"; the word for 12 is "two to the foot"; for 15, "a whole foot"; for 16, "one to the other foot"; the word for 20 is "one Indian"; for 40, "two Indians"; etc.

$3x = 50$. The formal solution is $x = \frac{50}{3}$; the possibility of its interpretation depends upon the character of the special problem.

The Egyptians used fractions before 1700 B.C.* and resolved them into sums of unit fractions. Thus $\frac{2}{5}$ was written $\dot{3}\ \dot{15}$, which meant $\frac{1}{3} + \frac{1}{15}$. By the Babylonians, $\frac{5}{8}$ was written 37 30, which meant $\frac{37}{60} + \frac{30}{60^2}$. Instead of fractions, the Greek geometers used the ratio of two numbers or two magnitudes.

For the introduction of **negative** numbers (as well as the decimal positional system and the symbol 0), algebra is indebted to the early mathematicians of India (between 500 and 600 A.D.). We now find it convenient to write $-15°$ for $15°$ below zero; -100 ft. for 100 feet below sea-level, thereby abbreviating our map notations. The determination of a number x such that $b + x = c$ leads to the algebraic solution $x = c - b$. If b exceeds c, the result $c - b$ is a negative number. Such a result would be excluded if the problem were to find how many feet of rope must be added to a rope b feet long to make a rope c feet long. But the negative result leads us to restate the problem so that the required c-foot rope is seen to be obtained by cutting off $b - c$ feet of rope from the b-foot rope.

In this connection, we note the Roman notations IV for 4, VI for 6, IX for 9, XI for 11, which seem to have been of Etruscan origin.

The term **rational number** includes positive and negative integers and fractions. All other numbers are called **irrational.**

The solution of equations of the form $x^n = A$, where A is a rational number and n a positive integer, introduces two classes of irrational numbers. Thus, for $n = 2$, and A a positive integer not the square of a rational number, the square root of A, denoted by the symbol $\sqrt{A}$, is an irrational number called a **quadratic surd.** Similarly, $\sqrt[3]{2}$, $\sqrt[3]{-4}$, $\sqrt[3]{A}$, A not the cube of a rational number, are **surds of the third order.** In general, $\sqrt[n]{A}$, where A is positive if n is even, is a **surd of order** n. The second class of irrational numbers are defined by the symbols $\sqrt[n]{A}$, where A is negative and n even (§ 4).

---

* Rhind papyrus, "Directions for Attaining to the Knowledge of All Dark Things."

While the equation $x^2 = 80$ possesses the formal solutions $x = \pm \sqrt{80}$ (the positive root is designated $\sqrt{80}$, the negative $-\sqrt{80}$), the possibility of the interpretation of one or both results depends upon the character of the particular problem. It is possible to form a square of area 80 square feet, but impossible to arrange 80 square blocks of equal size in the form of a square and yet preserve the form of each block.

2. The fact that surds really exist as such may be illustrated by showing that $\sqrt{2}$ is not expressible as a rational number. If we take the side of a square as the unit of length, the diagonal is of length $\sqrt{2}$. But it is proved in Geometry that the side and diagonal are incommensurable (see § 55). Hence 1 and $\sqrt{2}$ have no common measure. It follows that $\sqrt{2}$ is not equal to a rational number. For, if

$$\sqrt{2} = \frac{p}{q}, \tag{1}$$

then $\frac{1}{q}$ would be contained p times in $\sqrt{2}$ and q times in 1 and hence be a common measure of 1 and $\sqrt{2}$.

To give a purely algebraic proof, suppose that equation (1) holds true, the fraction $\frac{p}{q}$ being in its lowest terms, so that p and q are integers having no common divisor. By squaring and multiplying by q^2, we get $2q^2 = p^2$, so that p^2 and therefore p is divisible by 2. Setting $p = 2r$, we get $q^2 = 2r^2$, so that q is divisible by 2. Then p and q have a common divisor 2, contrary to hypothesis.

3. But with the introduction of rational numbers and surds, we do not meet all the demands which are made upon a number system. We are led in Geometry to consider the number π which expresses the ratio of the circumference to the diameter of a circle and to approximate its value by considering the perimeters p_6 and P_6 of an inscribed and a circumscribed regular hexagon, the perimeters p_{12} and P_{12} of an inscribed and a circumscribed regular polygon of 12 sides, etc. From the results, true to four decimal places, for a circle of unit diameter:

$$\begin{aligned} p_6 &= 3 \quad\ \ , & p_{12} &= 3.1058, \ \ldots, & p_{384} &= 3.1415, \ \ldots \\ P_6 &= 3.4641, & P_{12} &= 3.2153, \ \ldots, & P_{384} &= 3.1416, \ \ldots \end{aligned}$$

we obtain a succession of numbers between each pair of which the value of π must lie. By proving that

$$\begin{aligned} p_6 < p_{12} < p_{24} < \ldots < p_{384} < \ldots < \pi, \\ P_6 > P_{12} > P_{24} > \ldots > P_{384} > \ldots > \pi, \end{aligned}$$

and that the difference $P_n - p_n$ can be made to differ from zero as little as we please by sufficiently increasing the number of sides n, we have pointed out to us, with as great a degree of precision as we may desire, a certain limit, which we take as the value of π.

In an analogous manner, we can define the number $\sqrt{2}$ by means of two sequences of *rational* numbers,

$$\begin{array}{lllllll} 1, & 1.4, & 1.41, & 1.414, & 1.4142, & 1.41421, & \ldots \\ 2, & 1.5, & 1.42, & 1.415, & 1.4143, & 1.41422, & \ldots \end{array}$$

By the arithmetical process for the extraction of a square root, we find that the value of $\sqrt{2}$ lies between each pair of corresponding numbers in the sequences.

In general, two such sequences of rational numbers proceeding by a given law are said to define, by a limiting process, a **number**.* The value of the number may be determined to as great a degree of approximation as may be desired. All such numbers as well as all rational numbers are called **real numbers**.

---

* The above sequences which defined the number π can evidently be replaced by sequences of rational numbers related to the p_n and P_n.

4. An even root of a negative number is called an **imaginary quantity**. Thus $\sqrt{-1}$, $\sqrt[4]{-2}$, $\sqrt[6]{-1}$ are imaginaries. Unlike surds and other real numbers, an imaginary can not be expressed approximately in terms of rational numbers and hence has no interpretation in strictly arithmetical problems. By the introduction of imaginaries, we may give a formal solution of the equations $x^2 = -1$, $x^2 = -2$, and, indeed, of every quadratic equation. By extending the system of all real numbers by the introduction of the quantity $\sqrt{-1}$, we obtain the quantities $a + b\sqrt{-1}$, where a and b are arbitrary real numbers. We shall see that the system of these **complex quantities** $a + b\sqrt{-1}$ forms a number system within which may be performed all algebraic operations including the solution of all algebraic equations, so that a further extension is unnecessary. The employment in algebra of imaginaries has therefore a great practical value in that the operations may be effected without the limitations otherwise necessary. To further justify this extension, we recall that negative, fractional, and surd numbers were introduced to enable us to give a formal solution of many simple problems which would otherwise have remained insolvable, and that the possibility of the interpretation of negative, fractional, or irrational results depends upon the nature of the particular problem.*

5. *If* $a + \sqrt{b} = c + \sqrt{d}$, *where* a, b, c, d *are rational numbers and* $\sqrt{b}$ *is irrational, then* $a = c$, $b = d$.

From $a - c + \sqrt{b} = \sqrt{d}$, we derive, after squaring,

$$2(a - c)\sqrt{b} = d - b - (a - c)^2.$$

Unless the coefficient $a - c$ is zero, we could express $\sqrt{b}$ as a rational number, contrary to assumption. Hence $a = c$, so that $b = d$.

---

* A possible interpretation of complex quantities is given in the Appendix. The instructor may prefer the illustration by means of operations which combine a rotation with a magnification. Thus -1 rotates through 180°, $\sqrt{-1}$ through 90°, $4 + 3\sqrt{-1}$ magnifies five-fold and rotates. See Chrystal's *Algebra*, I, p. 239.

6. Let us attempt to extract the square root of $a + \sqrt{b}$, whe $\sqrt{b}$ is a true surd and a is rational. We seek a result of the for $\sqrt{\alpha} + \sqrt{\beta}$ in which α and β are rational. Setting

$$\sqrt{a + \sqrt{b}} = \sqrt{\alpha} + \sqrt{\beta},$$

and squaring, we find that

$$a + \sqrt{b} = \alpha + \beta + 2\sqrt{\alpha\beta}.$$

By the above theorem, we may equate the rational parts and al the irrational (surd) parts. Hence

$$a = \alpha + \beta, \qquad b = 4\alpha\beta.$$

Then $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = a^2 - b$, so that

$$\alpha = \tfrac{1}{2}(a + \sqrt{a^2 - b}), \qquad \beta = \tfrac{1}{2}(a - \sqrt{a^2 - b}).$$

By assumption α and β are rational. *Hence the square root o* $a + \sqrt{b}$ *is expressible as a sum of two quadratic surds* $\sqrt{\alpha} + \surd$ *if, and only if,* $a^2 - b$ *is the square of a rational number.*

For example, if $a = 6$, $b = 20$, $a^2 - b$ is the square of Hence $a + \sqrt{b} = 6 + \sqrt{20}$ is the square of $\sqrt{\alpha} + \sqrt{\beta} = \sqrt{5} +$

When the problem is solvable, it may usually be done by inspection as follows. Put the expression $a + \sqrt{b}$ into the form $m + 2\sqrt{n}$ by taking $m = a$, $n = b/4$. The required root is $\sqrt{\alpha} + \sqrt{\beta}$, where $\alpha + \beta = m$, $\alpha\beta = n$.

Thus $\qquad 6 + \sqrt{20} = 6 + 2\sqrt{5} = (1 + \sqrt{5})^2,$

since $\qquad 1 + 5 = 6, 1 \cdot 5 = 5.$

Thus $\qquad 16 + 6\sqrt{7} = 16 + 2\sqrt{63} = (\sqrt{7} + \sqrt{9})^2,$

since $\qquad 7 + 9 = 16, 7 \cdot 9 = 63.$

7. Denote by i the symbol $\sqrt{-1}$. Then $+i$ and $-i$ are the roots of $x^2 = -1$. By the symbol $\sqrt{-c}$, where c is positive, we shall mean $\sqrt{-1}\sqrt{c} = i\sqrt{c}$, where $\sqrt{c}$ denotes the positive square root of c. Thus

$$i^2 = -1, \quad i^3 = -i, \quad i^4 = +1, \quad i^{2n} = (-1)^n, \quad i^{2n+1} = i(-1)^n.$$

If c and d are any two positive real quantities,

$$\sqrt{-c}\sqrt{-d} = i\sqrt{c}\cdot i\sqrt{d} = -\sqrt{c}\sqrt{d} = -\sqrt{cd}.$$

In introducing the equation $x^2 = -c$ as an equation to be solved by algebra, we are tacitly assuming that x may be combined with itself and with real numbers according to the laws of algebra for the combination of real numbers, so that $d\sqrt{-c} = \sqrt{-c}\,d$, $d + \sqrt{-c} = \sqrt{-c} + d$, etc. In addition to these assumptions, we assume that complex quantities may be combined according to the laws holding for real numbers. Then

$$(a + bi) \pm (\alpha + \beta i) = (a \pm \alpha) + (b \pm \beta)i,$$
$$(a + bi)(\alpha + \beta i) = (a\alpha - b\beta) + (a\beta - b\alpha)i,$$
$$\frac{\alpha + \beta i}{a + bi} = \frac{(\alpha + \beta i)(a - bi)}{(a + bi)(a - bi)} = \frac{a\alpha + b\beta}{a^2 + b^2} + \frac{a\beta - b\alpha}{a^2 + b^2}i.$$

*Hence the sum, difference, product, or quotient of any two complex quantities is itself a complex quantity.**

In freeing the denominator of the above fraction from imaginaries, we used the multiplier $a - bi$, called the **conjugate** of the denominator $a + bi$. The sum and the product of two conjugate complex quantities are both real.

8. *The three cube roots of unity are*

$$1,\quad \omega = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3},\quad \omega^2 = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3},$$

so that $\omega^3 = 1,\quad 1 + \omega + \omega^2 = 0.$

The roots of $x^3 = 1$ are $x = 1$ and the two roots of

$$\frac{x^3 - 1}{x - 1} = x^2 + x + 1 = 0.$$

Completing the square in the quadratic equation, we get

$$x^2 + x + \tfrac{1}{4} = (x + \tfrac{1}{2})^2 = -\tfrac{3}{4}.$$

Hence $x = -\frac{1}{2} \pm \sqrt{-\frac{3}{4}}$. Setting $-\frac{1}{2} + \frac{1}{2}\sqrt{-3} = \omega$, the second root is

$$-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3} = (-\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3})^2 = \omega^2.$$

9. *In an equation between two complex quantities, the real parts are equal and also the imaginary parts.*

---

* The complex quantity $a + bi$ is real if $b = 0$.

Let $a + bi = \alpha + \beta i$, where a, b, α, β are real numbers. Then

$$a - \alpha = (\beta - b)i.$$

Upon squaring, we find that the number $(a - \alpha)^2$, which is positive or zero, must equal the number $-(\beta - b)^2$, which is negative or zero, so that each must be zero. Hence $a = \alpha$, $b = \beta$.

In particular, if $a + bi = 0$, then $a = 0$, $b = 0$.

10. *The square root of any complex quantity may always be expressed as a complex quantity.**

Let the given complex quantity be $a + bi$, where a and b are real and $i = \sqrt{-1}$. We seek real numbers x and y which will make

$$\sqrt{a + bi} = x + yi.$$

Squaring, $$a + bi = x^2 - y^2 + 2xyi.$$

Equating the real parts and also the imaginary parts,

$$x^2 - y^2 = a, \quad 2xy = b.$$

Then $$(x^2 - y^2)^2 + 4x^2y^2 = (x^2 + y^2)^2 = a^2 + b^2.$$

Since x and y are to be real, $x^2 + y^2$ must be positive. Hence

$$x^2 + y^2 = \sqrt{a^2 + b^2} \qquad \text{(positive square root).}$$

Having the sum and difference of x^2 and y^2, we derive

$$x^2 = \frac{\sqrt{a^2 + b^2} + a}{2}, \quad y^2 = \frac{\sqrt{a^2 + b^2} - a}{2}.$$

Since $\sqrt{a^2 + b^2}$ is positive and greater than a, the expressions for x^2 and y^2 are positive, so that *real* values of x and y may be determined by extracting the square roots of positive quantities. Since $2xy = b$, the sign of y is determined as soon as the sign of x is chosen. Hence there are always two and only two square roots of $a + bi$.

For example, to find the square roots of $5 - 12i$, we have

$$x^2 - y^2 = 5, \quad 2xy = -12,$$

whence $x^2 + y^2 = 13$, $x^2 = 9$, $y^2 = 4$. The square roots are $\pm(3 - 2i)$.

---

* Contrast with the theorem of § 6. The extraction of higher roots of complex quantities is done very simply in terms of trigonometric ratios [see Appendix].

The inspection method of § 6 may be extended to find the square roots of certain complex numbers $a + b\sqrt{-1}$. Thus for $-3 + 4\sqrt{-1}$, set

$$\sqrt{-3 + 2\sqrt{-4}} = \sqrt{x} + \sqrt{-y} \quad (x \text{ and } y \text{ positive}).$$

$$\therefore \quad -3 + 2\sqrt{-4} = x - y + 2\sqrt{-xy}.$$

$$\therefore \quad x - y = -3, \quad xy = 4 \qquad (\text{by § 9}).$$

Hence $x = 1$, $y = 4$, so that one square root of $-3 + 4\sqrt{-1}$ is $1 + 2\sqrt{-1}$.

EXERCISES.

Express with rational denominators

1. $\dfrac{2 - \sqrt{3}}{2 + \sqrt{3}}$. 2. $\dfrac{\sqrt{5} + 3}{\sqrt{5} - 2}$. 3. $\dfrac{\sqrt{3} + \sqrt{5}}{1 - \sqrt{15}}$.

4. $\dfrac{\sqrt{3} + \sqrt{5} + \sqrt{10}}{\sqrt{3} - \sqrt{5} + \sqrt{10}}$. 5. $\dfrac{1 + \sqrt{6}}{\sqrt{2} - \sqrt{3} + \sqrt{5}}$. 6. $\dfrac{3 + \sqrt{-5}}{2 + \sqrt{-1}}$.

7. $\dfrac{3 + 5\sqrt{-1}}{2 - 3\sqrt{-1}}$. 8. $\dfrac{a + b\sqrt{-1}}{a - b\sqrt{-1}}$.

9. Approximately, $\sqrt{2} = 1.4142$, $\sqrt{3} = 1.7320$; find $\dfrac{1}{\sqrt{2} - \sqrt{3}}$, $\dfrac{1 + \sqrt{2}}{1 - \sqrt{3}}$.

10. Simplify $\sqrt{45} + \sqrt{20} + 3\sqrt{5}$, $7\sqrt{24} - 2\sqrt{\tfrac{1}{8}}$, $\sqrt{15} \div \sqrt[4]{25}$.

11. Simplify $7\sqrt{-12} - 2\sqrt{-27}$, $\sqrt{-3} \times \sqrt{-48}$, $\sqrt{-9} \div \sqrt{-16}$

Express in terms of surds the square root of

12. $12 - 6\sqrt{3}$. 13. $28 - \sqrt{300}$. 14. $16 - 8\sqrt{3}$.

15. $29 + 6\sqrt{22}$. 16. $75 + 12\sqrt{21}$. 17. $47 - 4\sqrt{33}$.

18. $a + b + \sqrt{2ab + b^2}$. 19. $1 + a^2 + \sqrt{1 + a^2 + a^4}$.

Express in the form $a + b\sqrt{-1}$, a and b real, the square root of

20. $-11 + 60\sqrt{-1}$. 21. $-47 + \sqrt{-192}$.

22. $-20 + 48\sqrt{-1}$. 23. $-7 + 24\sqrt{-1}$.

24. $c^2 - d^2 - 2\sqrt{-c^2d^2}$. 25. $4cd + 2(c^2 - d^2)\sqrt{-1}$.

26. Prove, as in § 2, that $\sqrt{7}$ and $\sqrt[3]{4}$ are not expressible as rational numbers.

27. Prove that $1 + \sqrt{2}$ is not a surd by showing that an equation

$$(1 + \sqrt{2})^n = r = \text{a rational number}$$

would require $(1 - \sqrt{2})^n = r$, whereas the two equations are contradictory since $1 + \sqrt{2} > 1 - \sqrt{2}$.

CHAPTER II.

EXPONENTS; LOGARITHMS.

11. If m is a positive integer, a^m denotes the product of m factors each of which is a. Similarly, if n is a positive integer, $a^n = a \,.\, a \ldots a$, to n factors. Hence $a^m \,.\, a^n = a \,.\, a \ldots a$ to $m + n$ factors $= a^{m+n}$. We may therefore state, for the case of positive integral indices m, n,

The First Law of Indices. *The index of the product of two powers of the same quantity is the sum of the indices of the factors:*

(1) $$a^m \,.\, a^n = a^{m+n}.$$

As a corollary, we derive the formula

$$a^m \,.\, a^n \,.\, a^p \ldots . \, a^q = a^{m+n+p+\cdots+q}.$$

12. For the division of two positive integral powers of a, $a \neq 0$,

$$\frac{a^m}{a^n} = \frac{a \,.\, a \,.\, a \ldots a \quad (\text{to } m \text{ factors})}{a \,.\, a \,.\, a \ldots a \quad (\text{to } n \text{ factors})}$$

$$= a \,.\, a \ldots a \quad (\text{to } m - n \text{ factors})$$

if $m > n$. We may state, for m and n positive integers, $m > n$,

The Second Law of Indices. *The index of the quotient of two powers of the same quantity is the excess of the index of the numerator over the index of the denominator:*

(2) $$a^m / a^n = a^{m-n} \qquad (a \neq 0,\ m > n).$$

13. If m and n are positive integers, we have, by definition,

$$(a^m)^n = a^m \,.\, a^m \ldots a^m \quad (\text{to } n \text{ factors})$$

$$= a^{m+m+\cdots+m} = a^{mn} \quad (\text{by } \S\, 11).$$

Hence, for positive integral indices, we may state

The Third Law of Indices. *The index of the nth power of a^m is he product of the indices m and n:*

$$(a^m)^n = a^{mn}. \qquad \text{3)}$$

14. We next extend the use of the symbol a^n to cases in which is negative or fractional, assuming that such new symbols a^n will atisfy certain cases of the above *first* law of indices, and proceed o determine what meaning, if any, may be attached to the genralized symbols. It is later shown (§ 15) that the symbols, with he meanings thus obtained, satisfy the three laws of indices for m nd n any rational numbers. For this reason the interpretations f the symbols are justified and the desired permanence of the lgebraic laws is attained.

Consider the symbol $a^{\frac{1}{q}}$, where q is any positive integer. Since he symbol shall satisfy the first law of indices,

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \ldots \text{(to } q \text{ factors)} = a^{\frac{1}{q}+\frac{1}{q}+\frac{1}{q}+\ldots\text{(to } q \text{ terms)}} = a^1 = a.$$

Hence $a^{\frac{1}{q}}$ must be such that its qth power is a, that is,*

$$a^{\frac{1}{q}} = \sqrt[q]{a}.$$

Similarly, the symbol $a^{p/q}$, where p and q are positive integers, must be such that

$$a^{\frac{p}{q}} \cdot a^{\frac{p}{q}} \ldots \text{(to } q \text{ factors)} = a^{\frac{p}{q}+\frac{p}{q}+\ldots\text{(to } q \text{ terms)}} = a^p,$$

so that $a^{p/q}$ must be a qth root of a^p. Hence

$$a^{p/q} = \sqrt[q]{a^p}.$$

Since the symbol $a^{\frac{1}{q}}$ obeys the first law of indices, we find that

$$a^{\frac{1}{q}} \cdot a^{\frac{1}{q}} \ldots \text{(to } p \text{ factors)} = a^{\frac{p}{q}} = \left(\sqrt[q]{a}\right)^p.$$

---

* The radical sign is used to denote a particular root. Thus

$$\sqrt[2]{4} = +2, \; -\sqrt{4} = -2; \text{ hence } 4^{\frac{1}{2}} = +2.$$

Hence the two resulting values for $a^{p/q}$ must be equal,

$$\sqrt[q]{a^p} = (\sqrt[q]{a})^p,$$

a true theorem on radicals. Thus $\sqrt[3]{8^2} = (\sqrt[3]{8})^2 = 4$.

Consider the symbol a^0. By the first law of indices,

$$a^0 \cdot a^n = a^{0+n} = a^n,$$

so that $$a^0 = a^n/a^n = 1.$$

Hence, if a be any number different from zero, $a^0 = 1$.

Consider the symbol a^{-r}, where r is a positive integer or fraction, and assume that it may be combined with the symbols already defined in such a way that the first law of indices holds. Then

$$a^{-r} \cdot a^r = a^{r-r} = a^0 = 1,$$

$$a^{-r} = \frac{1}{a^r}.$$

Hence a^{-r} denotes the reciprocal of a^r.

As examples, $9^{\frac{3}{2}} = \sqrt{9^3} = \sqrt{3^6} = 3^3 = 27$,

$$4^{-\frac{3}{2}} = 1 \div 4^{3/2} = 1 \div \sqrt{4^3} = 1/8,$$

$$(-27)^{-\frac{2}{3}} = 1 \div (-27)^{\frac{2}{3}} = 1 \div (\sqrt[3]{-27})^2 = 1 \div (-3)^2 = 1/9.$$

15. We have been led to the following definitions:

(Def.) $$a^{\frac{p}{q}} = \sqrt[q]{a^p}, \quad a^0 = 1, \quad a^{-p/q} = 1 \div a^{p/q},$$

so that we have a definition for a^n for every rational number n. In order that these generalized symbols a^n, a^m shall prove to have practical value, it is essential that they shall obey the three formal laws of indices which have been proved to hold for positive integral exponents (§§ 11, 12, 13). In proving that this is the case, we make use of certain properties of radicals, namely:

(A) $$\sqrt[k]{t} = \sqrt[ks]{t^s},$$

(B) $$\sqrt[k]{t} \cdot \sqrt[k]{l} = \sqrt[k]{t \cdot l},$$

(C) $$\sqrt[k]{t} \div \sqrt[k]{l} = \sqrt[k]{t/l}.$$

I. *To prove that* $a^m a^n = a^{m+n}$ *when* m *and* n *are rational numbers.*

(1) Let m and n each be a positive fraction, including the case in which the denominator is unity. Then

$$
\begin{aligned}
a^{\frac{p}{q}}a^{\frac{r}{s}} &= \sqrt[q]{a^p}\sqrt[s]{a^r} && \text{(Def.)}\\
&= \sqrt[qs]{a^{ps}}\sqrt[qs]{a^{rq}} && \text{(by A)}\\
&= \sqrt[qs]{a^{ps}\cdot a^{rq}} && \text{(by B)}\\
&= \sqrt[qs]{a^{ps+rq}} && \text{(by first law, § 11)}\\
&= a^{\frac{ps+rq}{qs}} && \text{(Def.)}\\
&= a^{\frac{p}{q}+\frac{r}{s}}.
\end{aligned}
$$

(2) Let m be a positive fraction $\frac{p}{q}$ and n a negative fraction $-\frac{r}{s}$, and let $\frac{p}{q} > \frac{r}{s}$. Then $ps - rq$ is positive.

$$
\begin{aligned}
a^{\frac{p}{q}}a^{-\frac{r}{s}} &= \sqrt[q]{a^p} \div \sqrt[s]{a^r} && \text{(Def.)}\\
&= \sqrt[qs]{a^{ps}} \div \sqrt[qs]{a^{rq}} && \text{(by A)}\\
&= \sqrt[qs]{a^{ps} \div a^{rq}} && \text{(by C)}\\
&= \sqrt[qs]{a^{ps-rq}} && \text{(by second law, § 12)}\\
&= a^{\frac{ps-rq}{qs}} && \text{(Def.)}\\
&= a^{\frac{p}{q}+\left(-\frac{r}{s}\right)},
\end{aligned}
$$

so that $a^m a^n = a^{m+n}$ in this case.

(3) Let $m = \frac{p}{q}$, $n = -\frac{r}{s}$, with $\frac{p}{q} < \frac{r}{s}$. By case (2)

$$a^{\frac{r}{s}}a^{-\frac{p}{q}} = a^{\frac{r}{s}-\frac{p}{q}} \qquad \left(\text{since } \frac{r}{s} > \frac{p}{q}\right).$$

Taking the reciprocal of each member, we have by (Def.),

$$a^{-\frac{r}{s}}a^{\frac{p}{q}} = a^{-\left(\frac{r}{s}-\frac{p}{q}\right)} = a^{-\frac{r}{s}+\frac{p}{q}}.$$

(4) Let $m = -\frac{p}{q}$, $n = -\frac{r}{s}$, where $\frac{p}{q}, \frac{r}{s}$ are positive fractio Taking the reciprocal of the two equal expressions in case (1) a applying the definition of a negative exponent, we get

$$a^{-\frac{p}{q}}a^{-\frac{r}{s}} = a^{-\left(\frac{p}{q}+\frac{r}{s}\right)} = a^{-\frac{p}{q}-\frac{r}{s}}.$$

(5) Let finally m be rational and n be zero. By (Def.)

$$a^m \cdot a^0 = a^m \cdot 1 = a^m = a^{m+0}.$$

II. *To prove that* $a^m \div a^n = a^{m-n}$ *when* m *and* n *are rational.*

$$\begin{aligned} a^m \div a^n &= a^m \cdot \frac{1}{a^n} = a^m \cdot a^{-n} \qquad \text{(Def.)} \\ &= a^{m+(-n)} = a^{m-n}, \end{aligned}$$

since m and $(-n)$ are rational numbers, so that Theorem I applies.

III. *To prove that* $(a^m)^n = a^{mn}$ *when* m *and* n *are rational.*

(1) Let m be a positive fraction and n a positive integer. Then

$$\begin{aligned} \left(a^{\frac{p}{q}}\right)^n &= \sqrt[q]{a^p} \cdot \sqrt[q]{a^p} \ldots \text{(to } n \text{ factors)} && \text{(Def.)} \\ &= \sqrt[q]{a^p \cdot a^p \ldots \text{(to } n \text{ factors)}} && \text{(by B)} \\ &= \sqrt[q]{a^{pn}} && \text{(§ 11, corollary)} \\ &= a^{\frac{p}{q} \cdot n} && \text{(Def.)} \end{aligned}$$

(2) Let m and n be positive fractions. Then

$$\begin{aligned} \left(a^{\frac{p}{q}}\right)^{\frac{r}{s}} &= \sqrt[s]{\left(a^{\frac{p}{q}}\right)^r} && \text{(Def.)} \\ &= \sqrt[s]{a^{\frac{pr}{q}}} && \text{[by case (1)]} \\ &= a^{\frac{pr}{qs}} && \text{(Def.)} \end{aligned}$$

(3) Let m be a positive and n a negative number, integral or fractional. Then, using the definition of a negative exponent twice,

$$(a^m)^{-\nu} = \frac{1}{(a^m)^\nu} = \frac{1}{a^{m\nu}} = a^{-m\nu} \qquad \text{[by case (2)]}.$$

(4) Let m be negative and n positive. Then

$$(a^{-\mu})^n = \left(\frac{1}{a^\mu}\right)^n = \frac{1}{(a^\mu)^n},$$

since $\left(\frac{1}{w}\right)^n = \frac{1}{w^n}$ is true for positive integral n by inspection, and also for n a positive fraction in view of

$$\left(\frac{1}{w}\right)^{\frac{r}{s}} = \sqrt[s]{\left(\frac{1}{w}\right)^r} = \sqrt[s]{\frac{1}{w^r}} = \frac{1}{\sqrt[s]{w^r}} = \frac{1}{w^{r/s}}.$$

Next, $(a^\mu)^n = a^{\mu n}$, for μ and n positive, by case (2). Hence

$$(a^{-\mu})^n = \frac{1}{a^{\mu n}} = a^{-\mu n}$$

(5) Let, finally, m and n be both negative. Then

$$(a^{-\mu})^{-\nu} = \frac{1}{(a^{-\mu})^\nu} = \frac{1}{a^{-\mu\nu}} \qquad \text{[by case (4)]}$$

$$= a^{\mu\nu}. \qquad \text{(Def.).}$$

The laws of indices therefore hold when the exponents m and n are any rational numbers. As an exercise, the student may prove that, for any rational number n,

$$(ab)^n = a^n b^n, \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

By way of illustration of the laws of indices, we note that

$$a^{\frac{3}{4}} \cdot a^{-\frac{1}{3}} = a^{\frac{3}{4} - \frac{1}{3}} = a^{\frac{5}{12}}, \quad a^{\frac{3}{4}} \div a^{-\frac{1}{3}} = a^{\frac{13}{12}},$$

$$\left(a^{\frac{3}{4}}\right)^{-\frac{1}{3}} = a^{-\frac{1}{4}} = \frac{1}{\sqrt[4]{a}}, \quad a^{-2} \div a^{-6} = a^4,$$

$$a^{\frac{1}{2}}\, a^{-\frac{2}{3}}\, a^{\frac{1}{6}} = a^{\frac{1}{2} - \frac{2}{3} - \frac{1}{6}} = a^0 = 1.$$

EXERCISES.

Find the numerical value of

1. $8^{\frac{2}{3}}$. **2.** $16^{-\frac{3}{4}}$. **3.** $9^{-\frac{5}{2}}$. **4.** $\left(\frac{27}{8}\right)^{-\frac{1}{3}}$. **5.** $25^{-\frac{1}{2}}$. **6.** $100^{-\frac{3}{2}}$.

Express with positive indices

7. $3a^{-\frac{1}{2}}$. **8.** $2 \div x^{-2}$. **9.** $5 \div 3x^{-\frac{1}{2}}$.

10. $2 \div \sqrt{y^{-3}}$ **11.** $\sqrt[4]{x^7} \div \sqrt{x^{-3}}$.

Express by radical signs with positive indices

12. $a^{\frac{2}{3}}$. **13.** $a^{\frac{3}{2}}$. **14.** $a^{\frac{3}{4}}$. **15.** $a^{-\frac{3}{4}}$

16. $a^{-\frac{1}{2}}a^{\frac{3}{5}}$. **17.** $\frac{5a^{-3}}{a^{\frac{2}{3}}}$. **18.** $\sqrt[3]{a^{-2}}\sqrt{a^{-3}}$ **19.** $\frac{3x^{-1}}{x^{\frac{1}{4}}}$.

20. $\frac{8a}{b^{-\frac{2}{3}}}$. **21.** $\frac{3a^{-\frac{3}{4}}}{5a^{\frac{1}{4}}}$.

Simplify

22. $a^{\frac{1}{3}}a^{-\frac{2}{3}}(a^{3})^{-\frac{1}{3}}(a^{-1})^{-4}$. **23.** $a^{\frac{1}{3}}b^{\frac{2}{3}} \div a^{-\frac{2}{3}}b^{\frac{4}{3}}$.

24. $\left(a^{\frac{1}{2}}\sqrt[3]{b^{2}}\right)^{-3} \div \sqrt{b^{-4}\sqrt{a^{-2}}}$. **25.** $\frac{a^{-\frac{1}{2}}b^{-\frac{3}{4}}}{a^{\frac{1}{3}}b^{\frac{1}{4}}} \div \frac{a^{\frac{5}{6}}}{b^{-\frac{1}{6}}}$.

26. $2^{n}\cdot(2^{n-1})^{n} \div 2^{2n+1}\cdot 2^{2n-1}$ **27.** $(2^{n})^{n-1} \div (2^{n-1})^{n+1}$.

28. Multiply $x^{-\frac{2}{3}} + y^{-\frac{3}{2}}$ by $x^{-\frac{2}{3}} - y^{-\frac{3}{2}}$.

29. Multiply $a^{m} + a^{\frac{m}{2}} + 1$ by $a^{-m} + a^{-\frac{m}{2}} + 1$.

30. Divide $32x^{-6} + 12x^{-4} + 10x^{-2} - 12$ by $2x^{-2} - 1$.

31. Factor $a^{\frac{2}{3}} - b^{\frac{2}{3}}$, $x^{\frac{3}{2}} - y^{\frac{3}{2}}$, $x^{-3} - 27y^{-3}$.

32. Find the square root of $a^{4n} - 6a^{3n} + 5a^{2n} + 12a^{n} + 4$.

33. Find the square root of $4a - 4a^{\frac{5}{6}} + a^{\frac{2}{3}} + 2a^{\frac{7}{6}} + a^{\frac{5}{3}} - 4a^{\frac{4}{3}}$.

34. Solve $x - x^{\frac{1}{2}} - 30 = 0$; $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} + 2 = 0$.

35. Solve $x^{\frac{1}{3}} = 8x^{-\frac{1}{3}} + 2$; $x^{-4} + 16 = 8x^{-2}$.

16. What meaning, if any, may be attached to the new symbol $3^{\sqrt{2}}$, in which the index $\sqrt{2}$ is not rational? Since $\sqrt{2}$ can not be expressed as a fraction (Chapter I), the symbol $3^{\sqrt{2}}$ is of a character not previously considered. But $\sqrt{2}$ may be approximated by a decimal fraction to any desired degree of approximation. For example, $\sqrt{2}$ lies between 1.414 and 1.415; it lies between 1.41421 and 1.41422; etc. We have obtained a definite meaning for the expressions $3^{\frac{1414}{1000}}$ and $3^{\frac{1415}{1000}}$. It is natural to say that the value of $3^{\sqrt{2}}$ lies between the values of these two expressions. As a closer approximation, the value attached to $3^{\sqrt{2}}$ lies between

$$3^{\frac{141421}{100000}} \text{ and } 3^{\frac{141422}{100000}}.$$

Continuing the process of approximation, we obtain two fractions $\frac{m}{n}$ and $\frac{m+1}{n}$, which differ by an amount $\frac{1}{n}$ which can be made as small as we please, and such that $\sqrt{2}$ lies between the two fractions. By this limiting process, there is pointed out, with as great a degree of precision as we may desire, a so-called **limit**, which we define to be the value of $3^{\sqrt{2}}$. Similar definitions apply to the symbols $2^{\sqrt{3}}$, $a^{\sqrt[3]{n}}$ and, generally, to a^n where n is any real number. A series for a^n in ascending powers of n will be given in Chapter XVI; so that a more practical method is derived for the calculation of a^n.

If decimals correct to five places be used for $3^{\sqrt{2}}$ and $3^{\sqrt{3}}$, the product of the decimals will equal $3^{\sqrt{2}+\sqrt{3}}$ to an approximation correct to five places.* It follows that in numerical work we may employ the laws of indices when dealing with symbols like $3^{\sqrt{2}}$ by means of their approximate values. We make the assumption that the laws hold for the symbols $3^{\sqrt{2}}$, . . . , themselves.

LOGARITHMS.

17. By the first two laws of indices, two powers a^m and a^n of the same quantity a may be multiplied or divided by merely adding or subtracting the indices m, n. Now addition and subtraction are simpler operations to perform numerically than multiplication and division. It would thus appear desirable to be able to express all real numbers as powers of a single number a, called **the base**. Only positive numbers will be employed as bases.

For example, taking $a = 2$, we have

$$4 = 2^2,\ \ 8 = 2^3,\ \ 1 = 2^0,\ \ \tfrac{1}{2} = 2^{-1},\ \ \tfrac{1}{8} = 2^{-3},\ \ \sqrt{2} = 2^{\frac{1}{2}},\ \ \sqrt{8} = 2^{\frac{3}{2}}.$$

---

*In employing only five decimals, the fifth place in the product may vary by unity from the correct result for the fifth place *in the most unfavorable cases*. Thus, if only five places be retained in .972385, the square would be incorrect by more than .000009, which would be replaced by .00001 when only five places are retained.

But we do not find by inspection the power of 2 which will give certain numbers, for example, 3. By trial, we find that

$$2 < 2^{\frac{9}{6}} < 2^{\frac{38}{24}} < 3 < 2^{\frac{39}{24}} < 2^{\frac{10}{6}} < 2^2.$$

Now $2^{\frac{38}{24}} = 2.996$ to three decimal places. The required exponent is slightly greater than $\frac{38}{24}$, but is less than $\frac{39}{24}$. Just as the symbol $\sqrt[n]{3}$ was introduced to denote the real positive root of the equation $x^n = 3$, so now we employ the symbol $\log_2 3$, read logarithm 3 base 2, to denote the exponent x which makes $2^x = 3$.

Definition.—*The exponent of the power of the base a which gives rise to a number N is called the* **logarithm of N to the base** a; *symbolically,*

$$N = a^{\log_a N}. \tag{4}$$

Thus, for $a = 2$, the above relations give

$$2 = \log_2 4,\quad 3 = \log_2 8,\quad 0 = \log_2 1,\quad -1 = \log_2 \tfrac{1}{2},\quad -3 = \log_2 \tfrac{1}{8},$$
$$\tfrac{1}{2} = \log_2\sqrt{2},\quad \tfrac{3}{2} = \log_2\sqrt{8},\quad \tfrac{38}{24} < \log_2 3 < \tfrac{39}{24}.$$

Similarly, $1000 = 10^3$ gives $3 = \log_{10} 1000$;

$$.01 = 10^{-2} \text{ gives } -2 = \log_{10} .01.$$

18. For $N = a$, equation (4) gives $\log_a a = 1$. Also $a^0 = 1$ gives $\log_a 1 = 0$. Hence, *whatever the base may be, the logarithm of* 1 *is zero and the logarithm of the base is unity.*

Since logarithms are exponents, the three laws of indices (exponents) give rise to three corresponding properties of logarithms.

I. *The logarithm of a product equals the sum of the logarithms of its factors.*

Let the factors be N and M. By the definition (4) we have

$$N = a^{\log_a N},\quad M = a^{\log_a M}. \tag{5}$$

Hence, by the first law of indices, and the definition (4),

$$a^{\log_a N + \log_a M} = NM = a^{\log_a NM}.$$
$$\therefore\ \log_a NM = \log_a N + \log_a M.$$

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

Let the dividend be N and the divisor M. Applying the second law of indices to the identities (5), we get

$$\frac{N}{M} = a^{\log_a N - \log_a M}.$$

By the definition (4) applied to the number $\frac{N}{M}$, we get

$$\frac{N}{M} = a^{\log_a \frac{N}{M}}.$$

$$\therefore \quad \log_a \frac{N}{M} = \log_a N - \log_a M.$$

III. *The logarithm of the pth power of a number equals p times the logarithm of the number.*

Let N be the number. Raising the two members of (4) to the power p and applying the third law of indices, we get

$$N^p = a^{p \log_a N}.$$

But, by definition (4) applied to the number N^p,

$$N^p = a^{\log_a N^p}.$$

$$\therefore \quad \log_a N^p = p \log_a N.$$

19. Compare the logarithms to the base 2 which are given in § 17 with the logarithms of the same numbers to the base 8:

$\frac{2}{3} = \log_8 4$, $1 = \log_8 8$, $0 = \log_8 1$, $-\frac{1}{3} = \log_8 \frac{1}{2}$, $-1 = \log_8 \frac{1}{8}$, . . .

We observe, for each number N, that $\log_8 N = \frac{1}{3} \log_2 N$. But $\frac{1}{3} = \log_8 2$. Hence there is a constant multiplier $\log_8 2$ in passing from logarithms to the base 2 to logarithms to the base 8.

The student may readily verify the results in the table

Number N	$\text{Log}_{10} N$	$\text{Log}_{100} N$	$\text{Log}_{10^{\frac{1}{3}}} N$
10,000	4	2	12
1,000	3	3/2	9
100	2	1	6
10	1	1/2	3
1	0	0	0
0.1	− 1	− 1/2	− 3
0.01	− 2	− 1	− 6
0.001	− 3	− 3/2	− 9

For the numbers N of the table, $\log_{100} N = \frac{1}{2} \log_{10} N$; the multiplier which converts logarithms to the base 10 into logarithms to the base 100 is $\frac{1}{2} = \log_{100} 10$. Similarly, a logarithm to the base 10 must be multiplied by $3 = \log_{10^{\frac{1}{3}}} 10$ to give the logarithm of the same number to the base $10^{\frac{1}{3}}$.

In general, suppose that the first base is a and that the second base is b. By the definition (4),

$$N = a^{\log_a N}, \quad N = b^{\log_b N}, \quad a = b^{\log_b a}.$$

$$\therefore \quad b^{\log_b N} = a^{\log_a N} = (b^{\log_b a})^{\log_a N}.$$

Applying the third law of indices, we get

$$\log_b N = \log_a N \,.\, \log_b a.$$

To pass from a table of logarithms to the base a *to a table of logarithms to the base* b, *we employ the constant multiplier* $\log_b a$, *the logarithm of the old base with respect to the new base.*

EXERCISES.

1. Find $\log_2 32$, $\log_4 32$, $\log_4 2\sqrt{8}$, $\log_4 \sqrt[5]{64}$, $\log_5 5\sqrt{5}$.
2. $\log_c b \,.\, \log_b N = \log_c a \,.\, \log_a N$.
3. $\log_a N \div \log_a M = \log_b N \div \log_b M$.
4. Express $\log_{10} \frac{4}{9}$, $\log_{10} 60$, $\log_{10} 450$ in terms of $\log_{10} 2$ and $\log_{10} 3$.
5. Solve $5^{2x} - 5^{x+1} + 6 = 0$ in terms of $\log_{10} 2$ and $\log_{10} 3$.
6. Simplify $\log \left(\sqrt[3]{144}\sqrt{108} \div \sqrt[5]{1728}\sqrt[4]{56} \right)$.

20. Logarithms to the base 10 are known as **Common Logarithms*** and are used in numerical calculations. The chief advantage of the base 10 lies in the fact that from a known logarithm of a number N may be derived by inspection the logarithms of all numbers differing from N only in the position of the decimal point. Thus, if we know that $\log_{10} 2.23 = .3483$, then $\log_{10} 22.3 = \log_{10} 10 + \log_{10} 2.23 = 1.3483$, and $\log_{10} 2230 = 3.3483$. The decimal part .3483 common to the three logarithms is called the **mantissa**. The integral part of a logarithm is called the **characteristic**; it depends upon the position of the decimal point in the given number. Thus $\log_{10} 2.23$, $\log_{10} 22.3$, $\log_{10} 2230$ have respectively the characteristics 0, 1, 3.

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* Introduced in 1615 by Briggs (1556–1630), a contemporary of Napier (1550–1617), the inventor of logarithms. For a history of logarithms see the article by Glaisher on *Logarithms* in the Encyclopædia Britannica.

21. That a shifting of the decimal point in any number N does not alter the mantissa of $\log_{10} N$ follows from the fact that the shifting of the decimal point p places to the right multiplies N by 10^p, the shifting q places to the left divides N by 10^q. But

$$\log_{10}(N \times 10^p) = \log_{10} N + p \log_1 10 = p + \log_{10} N,$$
$$\log_{10}(N \div 10^q) = \log_{10} N - q \log_{10} 10 = -q + \log_{10} N.$$

In determining the mantissa we may therefore disregard the decimal point in the number. Thus, for the number 22.3, we use the sequence of digits 223 in looking for the mantissa in the Table of Logarithms given below. In taking the mantissa from the Table, the decimal point is to be placed before the digits found. Hence for $\log_{10} 22.3$ the mantissa is .3483; by inspection, 22.3 lies between 10^1 and 10^2, so that the characteristic is 1. Hence $\log_{10} 22.3 = 1.3483$.

22. We may determine the characteristic of the logarithm of any number N by inspection. The place immediately to the left of the decimal point is known as the **units place.** If there be digits to the *left* of units place, let their number be p. Then $N = M \times 10^p$, where M is a number whose first digit is in units place, so that M lies between 10 and $1 = 10^0$. The characteristic of $\log_{10} M$ is evidently zero and therefore that of $\log_{10} N$ is p. But if the first significant digit in N is in the nth place to the right of units place, then $N = M \div 10^n$, M being defined as before. Thus, if $N = 0.0024$, $n = 3$ and $M = 2.4$. Then the characteristic of $\log_{10} N$ is $-n$. We may combine the results to give the theorem:

The characteristic of the logarithm of a number is $+n$ if the first significant digit lies n places to the left of units place, but is $-n$ if the first significant digit lies n places to the right of units place.

EXAMPLE. Find the logarithms base 10 of 2400, .24, and .0024.

From the Table, we get $\log 2.4 = .3802$. Hence

$$\log 2400 = 3.3802, \quad \log 0.24 = \bar{1}.3802, \quad \log 0.0024 = \bar{3}.3802.$$

As in this example, the negative sign belonging to the characteristic is written above the latter. Otherwise, -1.3802 would be taken to mean that both 1 and the decimal .3802 were negative, whereas the mantissa is understood to be always positive. Hence $\bar{1}.3802$ denotes $-1+0.3802$. If we desired to find $\log\sqrt{.24}$, we should divide $\log .24 = \bar{1}.3802$ by 2. To do this, we write

$$\tfrac{1}{2}(\bar{1}.3802) = \tfrac{1}{2}(-2+1.3802) = -1+.6901 = \bar{1}.6901.$$

23. There follows on pages 24, 25 a table of Logarithms to the base 10 correct to four decimal places. For its origin see Chapter XVI.

24. *To find by the Table the logarithm of a given number.*[*]

Suppose the given number to consist of three digits, as 2.23. The first two digits 22 are to be found in the left-hand column headed *N*. On a line with these and in the column headed by the third digit 3, we find 3483. With the decimal point prefixed, the result .3483 is the mantissa of log 2.23. Since there are no places to the left of units place, we get

$$\log 2.23 = 0.3483.$$

Similarly, $\log 7 = \log 7.0 = 0.8451.$

If the given number consists of four or more digits, as 2.2325, we determine the logarithms of the numbers of three digits which are respectively less than and greater than the given number. Thus

$$\log 2.2300 = 0.3483$$
$$\log 2.2400 = 0.3502$$

The logarithm of 2.2325 lies between these two logarithms, whose difference is .0019. But 2.2325 exceeds the smaller number 2.2300 by $\frac{25}{100}$ of the difference of the two numbers. Hence log 2.2325 must exceed log 2.2300 by approximately $\frac{25}{100}$ of .0019, so that $\log 2.2325 = 0.3483 + .0005 = 0.3488$.[†]

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[*] Henceforth the base is supposed to be 10.

[†] The value .0005 is used for .000475, since the former is the decimal of four places nearest to the latter. It would be deceptive to retain more than four places, since the results from the Table are only true to four places, and the retention of six places would indicate our belief in the accuracy of the fifth and sixth places.

In Chapter XVI, we establish the principle here involved, namely, *for a relatively small increase in a number, its logarithm increases proportionally.* For example, from the Table:

$$\log 9.34 = .9703,\quad \log 9.35 = .9708,\quad \log 9.36 = .9713,$$
$$\log 9.37 = .9717,\quad \log 9.38 = .9722,\quad \log 9.39 = .9727,$$

the difference between adjacent logarithms being 5 in four cases and 4 in one case. The difference 1 in the numbers 934, 935, . . . , 939 is relatively small. On the contrary, the principle is not valid if applied to the numbers 100, 200, 300, whose differences are not relatively small. Thus, from the Table,

$$\log 100 = 2.0000,\quad \log 200 = 2.3010,\quad \log 300 = 2.4771.$$

25. *To find by the Table the number corresponding to a given logarithm.*

Let the given logarithm be 3.9779 and the corresponding number be N. Since the mantissa .9779 does not occur in the Table, we take those numbers in the body of the Table, viz., 9777 and 9782, which are just larger and just smaller than 9779, respectively. Then

$$\left.\begin{array}{l}\log 9500 = 3.9777\\ \log\ \ N\ \ = 3.9779\\ \log 9510 = 3.9782\end{array}\right\}\begin{array}{l}\left.\begin{array}{l}\\ \\ \end{array}\right\}.0002 = \text{difference}\\ .0005 = \text{difference}\\ \end{array}$$

Hence the difference between 9500 and N must equal $\frac{2}{5}$ of 10, the difference between 9500 and 9510. Hence $N = 9500 + 4 = 9504$.

If the given logarithm is $\overline{3}.1795$, the work is as follows:

$$\left.\begin{array}{l}\log .00151 = \overline{3}.1790\\ \log\ \ N\ \ \ = \overline{3}.1795\\ \log .00152 = \overline{3}.1818\end{array}\right\}\begin{array}{l}\left.\begin{array}{l}\\ \\ \end{array}\right\}.0005 = \text{difference}\\ .0028 = \text{difference}\\ \end{array}$$

Hence the difference between .00151 and N must equal $\frac{5}{28}$ of .00001, the difference between .00151 and .00152. Hence $N = .00151 + .000002 = .001512$ to *four significant places.* For $\frac{5}{28} \times .00001 = .00000178 +$, we take .000002 as the additive part to obtain N, being a decimal all of whose figures are reliable. Had we used a six-

TABLE OF FOUR-PLACE LOGARITHMS.

N	0	1	2	3	4	5	6	7	8	9
0	0000	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981

TABLE OF FOUR-PLACE LOGARITHMS.

N	0	1	2	3	4	5	6	7	8	9
50	6990	6998	7097	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

place table of logarithms, the additive part would have been retained in the form .00000178.

26. An equation in which the unknown quantity appears in an exponent is called an **exponential equation.** Such equations may usually be solved with the aid of logarithms.

EXAMPLE. Find the value of x in $3^{2x+1} = 2^{5x-4}$.

Taking the logarithm of each member of the equation, we get

$$(2x + 1) \log 3 = (5x - 4) \log 2.$$

$$\therefore \quad x(2 \log 3 - 5 \log 2) = -4 \log 2 - \log 3.$$

$$\therefore \quad x(\log 3^2 - \log 2^5) = -\log (2^4 \times 3).$$

$$\therefore \quad x = \frac{-\log 48}{\log 9 - \log 32} = \frac{-1.6812}{\bar{1}.4491} = \frac{1.6812}{.5509} = 3.0517,$$

to four decimal places. The denominator may be calculated otherwise, since it equals

$$\log \frac{9}{2^5} = \log \frac{9.5^5}{10^5} = \log (.28125) = \bar{1}.4491.$$

EXERCISES.

1. Find the common logarithms of 6752, 5.4321, 0.04682.

2. What numbers have the logarithms 2.1516, $\bar{1}.2222$, $\bar{4}.3333$?

3. Solve $3^{x-1} = 450$, $10^{1-2x} = 4^{3-4x}$, $2^{x+1}\,3^{3x-5} = 1000$.

Using logarithms, calculate to four significant places:

4. $\dfrac{8124 \times .00345}{.00069 \times 87.42}$. 5. $\sqrt[5]{12345.}$ 6. $(84.625)^{\frac{2}{3}}$.

7. $\sqrt[3]{45.24} \times \sqrt{.0004} \times \sqrt[7]{12345.}$ 8. 2^{100}. 9. $\sqrt[100]{5} \times \sqrt[10]{8765.}$

Solve by logarithms the simultaneous equations:

10. $3^{2x+3y} = 10$, $5^{x-6} = 125^{2y-3}$. 11. $3^{x+y} = 6^x$, $2^y = 9 \times 2^{x-1}$.

12. Find the number of digits in 25^{100}, in 2^{400}.

CHAPTER III.

FACTOR THEOREM; QUADRATIC EQUATIONS.

27. It is observed that $x - a$ is a factor of

$$x^2 - a^2, \quad x^2 - 3ax + 2a^2, \quad x^3 - ax^2 - x + a,$$

and that each expression vanishes when a is written in place of x. In general, if an expression E_x involving x has a factor $x - a$, and if Q_x is the quotient arising from the division of E_x by $x - a$, then

$$E_x = (x - a)Q_x,$$

so that E_x vanishes when a is written in place of x. The inverse of this theorem is called

The Factor Theorem. *If an integral expression involving x vanishes when a is written in place of x, it has the factor $x - a$.*

Divide the given expression E_x by $x - a$ until a remainder R is obtained which does not involve x. If Q_x be the quotient, then

$$E_x = (x - a)Q_x + R.$$

Substituting a for x in this identity, we get

$$E_a = R,$$

where E_a is the value of E_x when a is written for x. By hypothesis, E_a vanishes, so that the remainder R is zero. Hence E_x is exactly divisible by $x - a$.

Incidentally, we have also established

The Remainder Theorem. *Upon dividing an integral expression E_x by $x - a$, we obtain as the remainder E_a, the value of E_x when a is written for x.*

EXAMPLE 1. To verify that $x - a$ is a factor of $x^5 - a^5$, we put a for x in the expression and get $a^5 - a^5 = 0$.

EXAMPLE 2. $x - 1$ divides $x^3 - 2x^2 + 1$, since $1 - 2 + 1 = 0$.

28. We next apply the factor theorem to prove that, if n be a positive whole number, $x^n - y^n$ is always divisible by $x - y$, but that $x^n + y^n$ is never divisible by $x - y$. Putting y in place of x, the expressions become, respectively,

$$y^n - y^n = 0, \quad y^n + y^n \neq 0.$$

To prove that $x^n + y^n$ has the factor $x + y$ when n is odd, but not when n is even, we note that $x + y = x - (-y)$. Upon writing $-y$ in place of x, $x^n + y^n$ becomes $(-1)^n y^n + y^n$, which vanishes if n is odd, but not if n is even. Similarly, $x^n - y^n$ has the factor $x + y$ if n is even, but not if n is odd.

In particular, $r^n - 1$ is always divisible by $r - 1$. We have

$$\begin{aligned} r^2 - 1 &= (r - 1)(r + 1), \\ r^3 - 1 &= (r - 1)(r^2 + r + 1), \\ r^4 - 1 &= (r - 1)(r^3 + r^2 + r + 1). \end{aligned}$$

To establish the general formula

$$(1) \qquad r^n - 1 = (r - 1)(r^{n-1} + r^{n-2} + \ldots + r + 1),$$

we perform the multiplication on the right and obtain

$$(r^n + r^{n-1} + \ldots + r^2 + r) - (r^{n-1} + \ldots + r + 1) = r^n - 1.$$

Substituting $\frac{x}{y}$ for r in (1) and multiplying by y^n, we get

$$(2) \quad x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^n \quad).$$

Replacing y by $-y$ we obtain, *if n is odd,*

$$(3) \quad x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots - xy^{n-2} + y^{n-1}).$$

Replacing y by $-y$ in (2), we obtain, *if n is even,*

$$(4) \quad x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-} y^2 - \ldots + xy^{n-2} - y^{n-1}).$$

29. The preceding formulæ may be applied to find a factor which will rationalize any binomial surd. For example,

$$\frac{1}{\sqrt[3]{3} - 1} = \frac{\sqrt[3]{9} + \sqrt[3]{3} + 1}{(\sqrt[3]{3} - 1)(\sqrt[3]{9} + \sqrt[3]{3} + 1)} = \frac{\sqrt[3]{9} + \sqrt[3]{3} + 1}{3 - 1}.$$

If the given surd is $x - y$, where $x = \sqrt[p]{a}$ and $y = \sqrt[q]{b}$, then x^n and y^n are rational if n be a multiple of both p and q. Applying formula (2), we see that the rationalizing factor is

$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \ldots + xy^{n-2} + y^{n-1},$$

the rational product being $x^n - y^n$.

If the given surd is $x + y$, where x, y, n are defined as above, then, for n odd, formula (3) gives the rationalizing factor

$$x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots - xy^{n-2} + y^{n-1},$$

and the rational product $x^n + y^n$; while for n even, the rational product is $x^n - y^n$ and, by formula (4), the rationalizing factor is

$$x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots + xy^{n-2} - y^{n-1}.$$

For example, to rationalize $\sqrt{7} + \sqrt[3]{5}$, the rationalizing factor is $(\sqrt{7})^5 - (\sqrt{7})^4\sqrt[3]{5} + (\sqrt{7})^3\sqrt[3]{25} - (\sqrt{7})^2 . 5 + \sqrt{7} . 5\sqrt[3]{5} - 5\sqrt[3]{25}$ and the rational product is $(\sqrt{7})^6 - (\sqrt[3]{5})^6 = 7^3 - 5^2 = 318$.

EXERCISES.

Decompose into three or more real factors

1. $x^4 - y^4$. 2. $x^6 - y^6$. 3. $x^9 + y^9$. 4. $x^9 - y^9$.

Without actual division show that

5. $18x^{10} + 19x^5 + 1$ is divisible by $x + 1$.
6. $2x^4 - x^3 - 6x^2 + 4x - 8$ is divisible by $x^2 - 4$.
7. $x^4 - 3x^3 + 3x^2 - 3x + 2$ is divisible by $x^2 - 3x + 2$.

Without actual division find the remainder when

8. $x^4 - 3x^2 + x - 6$ is divided by $x + 3$.
9. $x^6 + 3xy^5 + y^6$ is divided by $x - 2y$.

Express with rational denominator

10. $\dfrac{1}{\sqrt{3} - \sqrt[3]{5}}$. 11. $\dfrac{1}{\sqrt[3]{2} + 1}$. 12. $\dfrac{1 + \sqrt[3]{4}}{1 - \sqrt[3]{4}}$. 13. $\dfrac{\sqrt{2}\ \sqrt[3]{9}}{\sqrt{2} + \sqrt[3]{3}}$.

14. $(x + y)^5 - x^5 - y^5$ and $(x + y)^7 - x^7 - y^7$ are divisible by $x^2 + xy + y^2$

30. A quadratic equation is an equation which involves the square, but no higher power, of the unknown quantity. When all the terms have been transposed to the left member of the equation, the quadratic equation has the form

$$ax^2 + bx + c = 0, \tag{5}$$

where a, b, c are known constants. If upon substituting for x a particular number x_1, the left member of (5) reduces to zero, so that the equation is satisfied, x_1 is called a **root** of the equation.

A quadratic equation (5) may often be solved by inspection by factoring its left member. Thus $x^2 + x - 12 = 0$ may be written $(x - 3)(x + 4) = 0$, so that the roots are $+ 3$ and $- 4$.

When the factors are not evident, we complete the square of the terms involving x. For example,

$$x^2 - 6x - 91 = 0$$

may be written, after adding and subtracting 9,

$$(x - 3)^2 - 100 = 0, \quad \{(x - 3) + 10\}\{(x - 3) - 10\} = 0.$$

Corresponding to the factors $x + 7$, $x - 13$, we obtain the roots $- 7$, 13.

31. To solve the general equation $ax^2 + bx + c = 0$, where $a \neq 0$, we divide by the coefficient a of x^2 and obtain the equation

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0. \tag{6}$$

Adding and subtracting the square of $\frac{1}{2}\frac{b}{a}$, we get

$$\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right) = 0.$$

Since any quantity is the square of its square root, the left member may be regarded as the difference of two squares. Factoring, we get

$$\left\{x + \frac{b}{2a} + \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\right\}\left\{x + \frac{b}{2a} - \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}\right\} = 0.$$

Each factor furnishes one of the two roots

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}. \tag{7}$$

In particular, we derive the factorization

$$(8)\quad ax^2+bx+c=a\left\{x+\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\right\}\left\{x+\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}\right\}.$$

The roots of the quadratic equation $ax^2+bx+c=0$ *are*

$$(9)\qquad \alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

According as b^2-4ac ***is positive, negative, or zero, the roots are real and unequal, imaginary and unequal, or real and equal.***

For the last case, $b^2=4ac$, formula (8) becomes

$$ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2.$$

32. Taking the sum and the product of the roots, we get

$$(10)\qquad \alpha+\beta=-\frac{b}{a},\quad \alpha\beta=\frac{(-b)^2-(b^2-4ac)}{4a^2}=\frac{c}{a}.$$

Comparing these results with the quadratic (6), we conclude that, in a quadratic equation *where the coefficient of x^2 is unity*, the **sum** of the roots equals the negative of the coefficient of x, the **product** of the roots equals the constant term (the term not involving x).

For example, in the above quadratic $x^2-6x-91=0$, having the roots -7 and 13, the sum of the roots equals $+6$, the product of the roots equals -91.

To give a second proof by a method applicable also to equations of higher degrees (Chapter XIX), consider the equation*

$$(x-\alpha)(x-\beta)\equiv x^2-(\alpha+\beta)x+\alpha\beta=0.$$

Since $x-\alpha$ vanishes for $x=\alpha$, the equation has a root α. Similarly, it has a root β. By inspection, the sum of the roots is the negative of the coefficient x and their product is the constant term.

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*By using the symbol $\equiv$, read *identically equal*, we emphasize the fact that we have here two different forms of the left member of the *equation*, the two members of which are connected by the sign $=$, read *equal*.

The quadratic equation whose roots are given may therefore be constructed as follows:

$$x^2 - (\text{sum of roots}).x + (\text{product of roots}) = 0.$$

Thus, the quadratic equation whose roots are -2 and 4 is

$$x^2 - (-2 + 4).x - 8 \equiv x^2 - 2x - 8 = 0.$$

EXAMPLE 1. If α and β denote the roots of $x^2 - px + q = 0$, find the values of $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$, $\alpha^4 + \beta^4$.

Since $\alpha + \beta = p$ and $\alpha\beta = q$, we have

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2q,$$
$$\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha^2\beta - 3\alpha\beta^2 = p^3 - 3qp,$$
$$\alpha^4 + \beta^4 = (\alpha^2 + \beta)^2 - 2\alpha^2\beta^2 = (p^2 - 2q)^2 - 2q^2 = p^4 - 4p^2q + 2q^2.$$

EXAMPLE 2. Determine the quadratic equation whose roots are the cubes of the roots of $x^2 - px + q = 0$.

If α and β are the roots of the latter, we have (Ex. 1)

$$\alpha^3 + \beta^3 = p^3 - 3pq, \quad \alpha^3\beta^3 = q^3.$$

The quadratic equation with the roots α^3, β^3 is therefore

$$y^2 - (p^3 - 3pq)y + q^3 = 0.$$

EXAMPLE 3. Determine the condition upon the coefficients of the equation $ax^2 + bx + c = 0$ in order that one root may be four times the other.

By formulæ (10), $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$. Since β is, by the condition imposed, equal to 4α, we must have

$$5\alpha = -\frac{b}{a}, \quad 4\alpha^2 = \frac{c}{a}.$$

Equating the resulting values for α^2, we get the condition

$$4b^2 = 25ac.$$

EXERCISES.

Form the equations whose roots are

1. $4, -5$.
2. $c + d, c - d$.
3. $0, 4$.
4. $\frac{1}{2}, -\frac{1}{3}$.
5. $2 + \sqrt{3}, 2 - \sqrt{3}$.
6. $-3 + \sqrt{7}, -3 - \sqrt{7}$.
7. $\sqrt{-1}, -\sqrt{-1}$.
8. $4 \pm \sqrt{-3}$.
9. Find the condition that one root of $ax^2 + bx + c = 0$ shall be (1) the double of the other; (2) the negative of the other.
10. If α and β are the roots of $x^2 - px + q = 0$, find the equation whose roots are (1) $\frac{\alpha^2}{\beta}$ and $\frac{\beta^2}{\alpha}$; (2) $\alpha^3\beta$ and $\alpha\beta^3$; (3) p and q; (4) $\alpha + \beta$ and $\alpha^2 + \beta^2$; (5) $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$; (6) $\alpha + \beta$ and $\frac{1}{\alpha} + \frac{1}{\beta}$.

Factor the expressions and derive the roots of the corresponding equations:

11. $x^2 - 53x + 150$ **12.** $x^2 - 35x - 200.$ **13.** $4x^2 - 18 + 21x.$

14. $x^2 + 2ax - b^2 - 2ab.$ **15.** $x^2 - 4ax + 3a^2.$ **16.** $x^4 - 5x^2 + 4.$

17. $x^2 - \frac{1}{x^2} - a^2 + \frac{1}{a^2}.$ **18.** $\frac{x^2}{x+1} + \frac{x+1}{x^2} - 2.$ **19.** $x^2 + \frac{100}{x^2} - 29$

33. Symmetry. An expression is said to be **symmetrical** with respect to two or more letters if they enter the expression in such a manner that it is unaltered when any two of the letters are interchanged.

Thus $2a + 2b$ and $a^3 + b^3$ are symmetrical with respect to a and b; $a + b + c$ and $ab + ac + bc$ are symmetrical with respect to a, b, c.

An expression like $ab^3 + bc^3 + ca^3$ possesses only a partial symmetry; it is unaltered when a is replaced by b, b by c, and c by a, but is altered if two letters are interchanged. It is said to possess **cyclo-symmetry.**

EXAMPLE 1. Factor the cyclo-symmetrical expression

$$S \equiv a^3(b - c) + b^3(c - a) + c^3(a - b).$$

Writing b in place of a, the expression becomes

$$b^3(b - c) + b^3(c - b) \equiv 0.$$

Hence, by the factor theorem, $a - b$ is a factor of S. Similarly, $b - c$ and $c - a$ are factors of S, results following also from the cyclo-symmetry of S. Since S is of the fourth degree in the letters a, b, c, there remains a fourth factor $la + mb + nc$ of the first degree in a, b, c, so that

$$\frac{S}{(a-b)(b-c)(c-a)} = la + mb + nc. \tag{11}$$

The numerator and denominator of the fraction both possess cyclo-symmetry and therefore also the fraction does. Hence

$$la + mb + nc \equiv lb + mc + na.$$

From this identity we derive $l = m = n$. To prove that $l = -1$, we observe that the term $-a^2(b - c)$ of the denominator of (11) must divide into the term $a^3(b - c)$ of S to give the term la of the quotient. Another method is to employ particular values, say $a = 0$, $b = 1$, $c = 2$, when the identity (11) becomes

$$-6/2 = m + 2n = 3l.$$

Hence $$S = -(a - b)(b - c)(c - a)(a + b + c).$$

EXAMPLE 2. Factor the cyclo-symmetrical expression

$$E \equiv (a - b)^5 + (b - c)^5 + (c - a)^5.$$

As in Example 1, we see that E is divisible by the product

$$P \equiv (a-b)(b-c)(c-a).$$

Since E and P possess cyclo-symmetry, the quotient E/P is a cyclo-symmetrical expression of the second degree. Set

$$E/P = l(a^2+b^2+c^2)+m(ab+bc+ca).$$

For $a=0$, $b=1$, $c=-1$, we find that $15=2l-m$. For $a=0$, $b=1$, $c=2$, we get $15=5l+2m$. Hence $l=5$, $m=-5$. Hence

$$E=5(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca).$$

EXERCISES.

Factor the following cyclo-symmetrical expressions:

1. $ab(a-b)+bc(b-c)+ca(c-a)$.
2. $a^4(b-c)+b^4(c-a)+c^4(a-b)$.
3. $a^2b^2(a-b)+b^2c^2(b-c)+c^2a^2(c-a)$.
4. $ab(a^3-b^3)+bc(b^3-c^3)+ca(c^3-a^3)$.
5. $a^4(b^2-c^2)+b^4(c^2-a^2)+c^4(a^2-b^2)$.
6. $a^2(b^2-c^2)^3+b^2(c^2-a^2)^3+c^2(a^2-b^2)^3$.

CHAPTER IV.

SIMULTANEOUS EQUATIONS; DETERMINANTS.

34. If we assume that the values of the unknown quantities $x, y, \ldots$ are the same in each of several equations, the equations are called **simultaneous equations.**

Consider the pair of simultaneous equations

$$(1) \qquad \begin{cases} 2x - 3y = -4, \\ 6x - 2y = 2. \end{cases}$$

To solve for x, we multiply the first equation by -2 and the second equation by $+3$ and add the resulting equations. Then

$$14x = 14, \quad x = 1.$$

In a similar manner, we find that $y = 2$.

35. Consider the general pair of simultaneous equations

$$(2) \qquad \begin{cases} a_1x + b_1y = c_1, \\ a_2x + b_2y = c_2. \end{cases}$$

To solve for x, we eliminate y between the equations. This may be done by multiplying the first equation by b_2 and the second by $-b_1$ and adding the resulting equations. Then

$$(a_1b_2 - a_2b_1)x = (c_1b_2 - c_2b_1).$$

Employing the respective multipliers $-a_2$ and a_1, we get

$$(a_1b_2 - a_2b_1)y = (a_1c_2 - a_2c_1).$$

The quantities in the parentheses are all of the same form. We therefore employ as an abbreviation the symbol

$$(3) \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv a_1b_2 - a_2b_1,$$

called a **determinant of the second order.** It equals the difference of the two cross-products. The above relations may now be written

$$(4)\qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}.$$

By division, we obtain the values x, y satisfying equations (2). In formulæ (4), x and y are each multiplied by the symbol $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ which is formed of the coefficients in the left members of equations (2). The determinant $\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ is derived from the preceding determinant by replacing the a's by c's. We may express our results by the following theorem:

For the simultaneous equations (2), *x times the determinant D of the coefficients of the left members equals the determinant obtained by substituting the constants c_1, c_2 in place of a_1, a_2 in the first column of D; y times D equals the determinant obtained by substituting the constants c_1, c_2 in place of b_1, b_2 in the second column of D.*

Applying the theorem to example (1), we get

$$\begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix} x = \begin{vmatrix} -4 & -3 \\ 2 & -2 \end{vmatrix}, \quad \begin{vmatrix} 2 & -3 \\ 6 & -2 \end{vmatrix} y = \begin{vmatrix} 2 & -4 \\ 6 & 2 \end{vmatrix}.$$

Evaluating these determinants by the definition (3), we get

$$14x = 14, \quad 14y = 28.$$

EXERCISES.

Solve by determinants the pairs of simultaneous equations:

1. $8x - y = 34,\quad x + 8y = 53.$

2. $3x + 4y = 10,\quad 4x + y = 9.$

3. $x - y = 5,\quad 5x - 4y = 40.$

4. $ax - by = c,\quad cx - ay = b.$

5. $x + y = 1,\quad ax + by = c.$

6. $ax + by = a^2,\quad bx - ay = ab.$

36. Consider the three simultaneous equations

$$(5)\qquad \begin{cases} a_1x + b_1y + c_1z = k_1, \\ a_2x + b_2y + c_2z = k_2, \\ a_3x + b_3y + c_3z = k_3. \end{cases}$$

If we can combine these equations in such a way that y and z shall be eliminated, the resulting relation will, if it contains x, determine the value of x. Multiplying the first equation by M_1, the second equation by M_2, and the third equation by M_3, and adding the resulting equations, we get

$$(6)\qquad (a_1M_1 + a_2M_2 + a_3M_3)x + (b_1M_1 + b_2M_2 + b_3M_3)y + (c_1M_1 + c_2M_2 + c_3M_3)z = k_1M_1 + k_2M_2 + k_3M_3.$$

We desire that the coefficients of y and z shall vanish. It will be shown that this will indeed happen if we take

$$(7)\qquad M_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix},\quad M_2 = -\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix},\quad M_3 = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

Substituting for these determinants their values as given by the definition (3), we find that

$$(8)\qquad b_1M_1 + b_2M_2 + b_3M_3 = b_1(b_2c_3 - b_3c_2) - b_2(b_1c_3 - b_3c_1) + b_3(b_1c_2 - b_2c_1) \equiv 0.$$

In a similar manner, $c_1M_1 + c_2M_2 + c_3M_3 = 0$. Hence (6) becomes

$$(9)\qquad (a_1M_1 + a_2M_2 + a_3M_3)x = k_1M_1 + k_2M_2 + k_3M_3.$$

For the coefficient of x, with the values of M_1, M_2, M_3 inserted from (7), we shall employ the following symbol:

$$(10)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}.$$

It is noticed that the symbol is formed of the coefficients of the left members of equations (5), the coefficients retaining the same relative positions in the symbol as in the equations. We may derive the right member of equation (5) from the coefficient of x by replacing a_1, a_2, a_3 by k_1, k_2, k_3, respectively. Hence equation (9) may be written in the form

$$(11)\qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}.$$

Such symbols are called **determinants of the third order.** If the determinant on the left is not zero, we obtain by division the value of x.

37. The preceding method may be seen to be a direct generalization of the method employed in § 35 for the solution of simultaneous equations in two unknown quantities. To solve equations (2) for x, we employed the multipliers b_2 and $-b_1$ [notice the alternation in sign, corresponding to the signs $+$, $-$, $+$ in the multipliers (7)]. We observe that, in the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, b_2 stands diagonally opposite to a_1 and that b_1 lies opposite to a_2; so that the multipliers used in solving for x lie opposite to the coefficients of x in the equations (2). For the three equations (5), the multipliers (7) used in solving for x lie opposite to the coefficients of x. Expressed more exactly, the first multiplier M_1 may be derived from the array of coefficients in the left members of (5) by erasing the row a_1, b_1, c_1 and the column a_1, a_2, a_3 containing a_1 and taking the determinant $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ of the remaining coefficients. The second multiplier M_2, apart from its sign, may be derived by erasing the row and the column containing a_2 and taking the determinant $\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$ of the remaining coefficients. Similarly, M_3 is gotten by erasing the row and the column containing a_3. The resulting determinants are called the **minors** of a_1, a_2, a_3, respectively.

In solving equations (2) for y, we employed the multipliers $-a_2$ and $+a_1$. Apart from their signs, they are, respectively, the minors of b_1 and b_2 (the coefficients of y in the given equations) in the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$. This suggests that, in solving equations (5) for y, we should employ as multipliers $-B_1$, $+B_2$, $-B_3$, where

$$(12) \qquad B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}, \quad B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}, \quad B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix},$$

which are, respectively, the minors of b_1, b_2, b_3 in the determinant (10) of the coefficients of the left members of (5). Then

$$(13) \qquad -a_1B_1+a_2B_2-a_3B_3=0, \quad -c_1B_1+c_2B_2-c_3B_3=0,$$

so that, for $M_1=-B_1$, $M_2=+B_2$, $M_3=-B_3$, equation (6) gives

$$(14) \qquad (-b_1B_1+b_2B_2-b_3B_3)y=-k_1B_1+k_2B_2-k_3B_3.$$

The coefficient of y, with the values of B_1, B_2, B_3 inserted from (12), may be expanded as follows:

$$(15) \qquad -b_1a_2c_3+b_1a_3c_2+b_2a_1c_3-b_2a_3c_1-b_3a_1c_2+b_3a_2c_1.$$

It is therefore equal to the determinant (10). The right member of (14) is derived from the coefficient of y by replacing b_1, b_2, b_3 by k_1, k_2, k_3, respectively. Hence (14) becomes

$$(16) \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}.$$

Incidentally, it was seen that the determinant on the left can be expanded according to the elements of the second column by multiplying b_1 by the negative of its minor B_1, b_2 by its minor B_2, b_3 by the negative of its minor B_3; and taking the sum of the resulting products.

Proceeding in a similar manner, we find that

$$(17) \qquad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}.$$

Hence to solve equations (5) *for any one of the unknown quantities x, y, z, we equate the product of the unknown and the determinant D of the coefficients of the left members of* (5) *to the determinant obtained from D by substituting the constants k_1, k_2, k_3 in place of the coefficients of that unknown.*

38. It follows also from the preceding developments that, if A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3 denote the minors of a_1, a_2, a_3,

b_1, b_2, b_3, c_1, c_2, c_3, respectively, in the determinant D, these relations hold:

$$\begin{array}{lll}
a_1A_1-a_2A_2+a_3A_3=D, & -a_1B_1+a_2B_2-a_3B_3=0, & a_1C_1-a_2C_2+a_3C_3=0, \\
b_1A_1-b_2A_2+b_3A_3=0, & -b_1B_1+b_2B_2-b_3B_3=D & b_1C_1-b_2C_2+b_3C_3=0, \\
c_1A_1-c_2A_2+c_3A_3=0, & -c_1B_1+c_2B_2-c_3B_3=0, & c_1C_1-c_2C_2+c_3C_3=D
\end{array}$$

The three expressions for D give three methods of expanding the determinant D. To expand D according to the elements of any column, we multiply each element by its minor, with the proper sign prefixed, and form the sum of the products. The proper sign depends upon the position of the element, as exhibited by the following scheme:

+	−	+
−	+	−
+	−	+

Compare the black and white spacing on a checker-board.

39. The second equation in the above set may be written

$$0=b_1A_1-b_2A_2+b_3A_3\equiv\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix}.$$

Hence a determinant vanishes if its first and second columns are alike. Similarly, the third equation gives

$$0=c_1A_1-c_2A_2+c_3A_3\equiv\begin{vmatrix} c_1 & b_1 & c_1 \\ c_2 & b_2 & c_2 \\ c_3 & b_3 & c_3 \end{vmatrix}.$$

Hence a determinant vanishes if its first and third columns are alike. Similarly, $-c_1B_1+c_2B_2-c_3B_3=0$ shows that a determinant vanishes if its second and third columns are alike. The last result also follows directly from the definition (10) of the determinant. Combining these results, we see that *a determinant of the third order vanishes if any two of its columns are alike.*

40. Upon expanding the three determinants

$$\begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & a_3 \end{vmatrix}, \quad \begin{vmatrix} a_1 & c_1 & b_1 \\ a_2 & c_2 & b_2 \\ a_3 & c_3 & b_3 \end{vmatrix}, \quad \begin{vmatrix} c_1 & b_1 & a_1 \\ c_2 & b_2 & a_2 \\ c_3 & b_3 & a_3 \end{vmatrix},$$

according to the elements b_1, b_2, b_3, we obtain in each case

$$b_1B_1 - b_2B_2 + b_3B_3 = -D.$$

But these determinants were obtained from D by interchanging two of its columns. *By the interchange of any two columns of a determinant of the third order, its value is changed in sign.*

The theorem of § 39 may be derived as a corollary to the last theorem. Indeed, if D has two columns alike, it is evidently unaltered upon interchanging them, whereas it must change in sign. From $D = -D$, we derive $D = 0$.

41. As an example, consider the simultaneous equations

$$\begin{aligned} x + y + z &= 11, \\ 2x - 6y - z &= 0, \\ 3x + 4y + 2z &= 0. \end{aligned}$$

Employing the definition (10), we find that

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -6 & -1 \\ 3 & 4 & 2 \end{vmatrix} = 11.$$

Hence formulæ (11), (16), and (17) give

$$11x = \begin{vmatrix} 11 & 1 & 1 \\ 0 & -6 & -1 \\ 0 & 4 & 2 \end{vmatrix}, \quad 11y = \begin{vmatrix} 1 & 11 & 1 \\ 2 & 0 & -1 \\ 3 & 0 & 2 \end{vmatrix}, \quad 11z = \begin{vmatrix} 1 & 1 & 11 \\ 2 & -6 & 0 \\ 3 & 4 & 0 \end{vmatrix}.$$

Each determinant has two zeros in one of its columns. It is therefore best to expand the determinant according to the elements of that column. The results are, respectively,

$$11\begin{vmatrix} -6 & -1 \\ 4 & 2 \end{vmatrix} = -11\cdot 8, \quad -11\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = -11\cdot 7, \quad 11\begin{vmatrix} 2 & -6 \\ 3 & 4 \end{vmatrix} = 11\cdot 26.$$

Hence $x = -8$, $y = -7$, $z = 26$.

42. Consider the simultaneous equations

$$(18)\qquad \begin{aligned} x + y + z &= 1, \\ ax + by + cz &= k, \\ a^2x + b^2y + c^2z &= k^2. \end{aligned}$$

The value of x is given by the relation

$$(19)\qquad \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} x = \begin{vmatrix} 1 & 1 & 1 \\ k & b & c \\ k^2 & b^2 & c^2 \end{vmatrix}.$$

Denote the coefficient of x by D. We may factor D without making use of its expanded form. If we substitute b for a in D, we obtain a determinant having its first and second columns alike, which therefore vanishes by § 39. Hence, by the factor theorem, $a - b$ is a factor of D. Similarly, $a - c$ and $b - c$ are factors of D. Since D is of the third degree in the letters a, b, c, we may write

$$(20)\qquad D \equiv m(a - b)(a - c)(b - c),$$

where m is a numerical factor not involving a, b, or c. For the particular values $a = 0$, $b = 1$, $c = -1$, the identity (20) becomes $2 = -2m$, whence $m = -1$. The same result follows from a comparison of the coefficients of a term, like bc^2, in the expanded forms of the two members of (20). Changing the sign of the factor $a - c$, we may state the result:

$$(21)\qquad \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a - b)(b - c)(c - a).$$

Changing a into k, we obtain $(k - b)(b - c)(c - k)$ as the value of the determinant in the right member of (19). Hence

$$x = \frac{(k - b)(c - k)}{(a - b)(c - a)}.$$

In a similar manner, it is found that

$$y = \frac{(k - c)(a - k)}{(b - c)(a - b)}, \quad z = \frac{(k - a)(b - k)}{(c - a)(b - c)}.$$

43. If, in place of the third equation (18), we take

$$a^4x + b^4y + c^4z = k^4,$$

the values of x, y, z depend upon determinants of the form

$$\Delta \equiv \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix}.$$

It follows as above that Δ is divisible by the product

$$P \equiv (a - b)(b - c)(c - a).$$

Since the degree of Δ in its expanded form is five, the quotient Δ/P is of the second degree in a, b, c. By the interchange of a and b, P is changed into $-P$ and Δ into $-\Delta$ (§ 40), so that Δ/P remains unaltered. Likewise Δ/P is unaltered by the interchange of a and c or by the interchange of b and c. Hence Δ/P is symmetrical in the three letters a, b, c, so that we may set

$$\Delta/P = m(a^2 + b^2 + c^2) + n(ab + ac + bc), \tag{22}$$

where m and n are numerical values independent of a, b, c. We may evaluate m and n by substituting in the identity (22) special values for a, b, c, such, however, that the denominator P does not vanish. For $a = 0$, $b = 1$, $c = -1$ we get $\frac{2}{2} = 2m - n$. For $a = 0$, $b = 1$, $c = 2$, we get $\frac{14}{2} = 5m + 2n$. Solving these two simultaneous equations, we find that $m = 1$, $n = 1$. Hence the above determinant may be factored thus:

$$\Delta = (a - b)(b - c)(c - a)(a^2 + b^2 + c^2 + ab + ac + bc).$$

EXERCISES.

Solve by determinants the following sets of simultaneous equations, cancelling the extraneous factors in 3, 4, 5, 6:

1. $x + y + z = 0,$
$x + 2y + 3z = -1,$
$x + 3y + 6z = 0.$

2. $x - 2y + z = 12,$
$x + 2y + 3z = 48,$
$6x + 4y + 3z = 84.$

3. $x + y + z = 1,$
$ax + by + cz = k,$
$a^3x + b^3y + c^3z = k^3.$

4. $x + y + z = 1,$
$a^2x + b^2y + c^2z = k^2,$
$a^3x + b^3y + c^3z = k^3.$

5. $ax + y + z = a - 3,$
$x + ay + z = -2,$
$x + y + az = -2.$

6. $ax + by + cz = 2a + b + c,$
$bx + cy + az = a + 2b + c,$
$cx + ay + bz = a + b + 2c.$

44. If $k_1 = 0$, $k_2 = 0$, $k_3 = 0$, equations (5) become

$$(23) \qquad a_1x+b_1y+c_1z = 0, \quad a_2x+b_2y+c_2z = 0, \quad a_3x+b_3y+c_3z = 0.$$

According to the general method of solution, we have

$$(24) \qquad Dx = 0, \quad Dy = 0, \quad Dz = 0, \quad D \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

If $D \neq 0$, we obtain only the evident set of solutions $x = y = z = 0$. But, if $D = 0$, the relations (24) impose no conditions upon x, y, z. In this case, equations (23) might have solutions other than the evident set $x = y = z = 0$. Let us seek the solutions for which $z \neq 0$, for example. The first and second equations (23) may then be written

$$a_1\frac{x}{z} + b_1\frac{y}{z} = -c_1, \quad a_2\frac{x}{z} + b_2\frac{y}{z} = -c_2.$$

For the unknown quantities $\frac{x}{z}$ and $\frac{y}{z}$ we obtain the relations

$$(25) \qquad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\frac{x}{z} = \begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}, \quad \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}\frac{y}{z} = \begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}.$$

It remains to inquire whether the resulting values of $\frac{x}{z}$ and $\frac{y}{z}$ satisfy the third relation (23), which may be written

$$(26) \qquad a_3\frac{x}{z} + b_3\frac{y}{z} + c_3 = 0.$$

To avoid fractions, we multiply (26) by $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ before substituting the values of $\frac{x}{z}, \frac{y}{z}$. The result is clearly

$$a_3\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix} + b_3\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

The left member is seen to be equal to D. Hence, if $D = 0$, the third condition (26) is satisfied by the values of $\frac{x}{z}$, $\frac{y}{z}$ given by (25). We must examine the special case when

$$C_3 \equiv \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$$

vanishes, since equations (25) then fail to determine $\frac{x}{z}$, $\frac{y}{z}$. If

$$A_3 \equiv \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

does not vanish, we divide the first and second equations (23) by x and obtain solutions for $\frac{y}{x}$, $\frac{z}{x}$, which satisfy the third equation if, and only if, $D = 0$. Similarly, if

$$B_3 \equiv \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

does not vanish, we obtain solutions $\frac{x}{y}$, $\frac{z}{y}$, if $D = 0$.

There remains the case $A_3 = 0$, $B_3 = 0$, $C_3 = 0$. Since a_1, b_1, c_1 are tacitly assumed to be not all zero, we suppose that $a_1 \neq 0$, for definiteness. We may then set $a_2 = pa_1$, whence $a_1b_2 - a_2b_1 = 0$ gives $b_2 = pb_1$, while $a_1c_2 - a_2c_1 = 0$ gives $c_2 = pc_1$. Hence the second equation (23) may be derived by multiplying the first equation by p. We need therefore only consider the first and third equations (23). As shown by the above method, these two equations determine the ratios of x, y, z except in the case

$$\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix} = 0, \quad \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} = 0, \quad \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} = 0.$$

In the latter case, the third equation is a consequence of the first. Since the equations then reduce to a single one, arbitrary values may be assigned to y and z; and the equation then determines x.

We may therefore state the following theorem:

The necessary and sufficient conditions that equations (23) have solutions x, y, z, not all zero, is that the determinant D shall vanish. If $D = 0$, the equations reduce to two equations or to one equation, according as the minors of the nine coefficients are not all zero, or are all zero. In the first case, the ratios $x : y : z$ are determined, so that one unknown is arbitrary; in the second case, two of the unknowns may be chosen arbitrarily.

Given, for example, the simultaneous equations

$$2x + 3y - 4z = 0, \quad 3x + 5y - z = 0, \quad 7x + 11y - 9z = 0,$$

the determinant D is zero, and relations (25) become

$$\frac{x}{z} = 17, \quad \frac{y}{z} = -10.$$

45. Theorem. *A determinant of the third order is not altered in value if we multiply the elements of any column by a constant and add the results to another column.*

Consider a determinant D and its expansion:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - a_2A_2 + a_3A_3.$$

Multiplying the b's by m and adding the products to the a's, we get

$$\begin{vmatrix} a_1 + mb_1 & b_1 & c_1 \\ a_2 + mb_2 & b_2 & c_2 \\ a_3 + mb_3 & b_3 & c_3 \end{vmatrix} = (a_1 + mb_1)A_1 - (a_2 + mb_2)A_2 + (a_3 + mb_3)A_3$$

$$= (a_1A_1 - a_2A_2 + a_3A_3) + m(b_1A_1 - b_2A_2 + b_3A_3) = D,$$

since the expression multiplied by m is the expansion of

$$\begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix},$$

which is zero by § 39. The theorem is therefore true for this case. To extend the proof to the general case when the elements of the ith column of D are multiplied by m and added to the jth column, the resulting determinant being called D_m, we observe that the jth column of D may be interchanged with the first column and the ith column interchanged with the second column of the new determinant, giving the final determinant $D' = \pm D$. Doing likewise with D_m, we get the determinant $D'_m = \pm D_m$. By the proof given above, $D'_m = D'$. Hence $D_m = D$.

As an example, consider the determinant

$$D \equiv \begin{vmatrix} 2 & 1 & 1 \\ 6 & 2 & 1 \\ 12 & 3 & 1 \end{vmatrix}.$$

Multiplying the second column by -2 and adding to the first,

$$D = \begin{vmatrix} 0 & 1 & 1 \\ 2 & 2 & 1 \\ 6 & 3 & 1 \end{vmatrix}.$$

Multiplying the third column by -1 and adding to the second,

$$D = \begin{vmatrix} 0 & 0 & 1 \\ 2 & 1 & 1 \\ 6 & 2 & 1 \end{vmatrix} = -2. \tag{27}$$

The aim is to obtain two zeros in one row or one column.

46. Theorem. *If the first, second, and third columns of a determinant of the third order be taken as the first, second, and third rows, respectively, of a new determinant, the resulting determinant equals the original determinant.*

Starting with the determinant D, given by formula (10), we form the determinant

$$D' \equiv \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \equiv a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & a_3 \\ c_2 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}.$$

The terms of D and D' which involve a_1 are seen to be equal by inspection. The terms of D' which involve a_2 are $-b_1a_2c_3 + c_1a_2b_3$, the same as in D. The terms of D' which involve a_3 are $b_1c_2a_3 - c_1b_2a_3$, the same as in D. Hence $D = D'$.

It follows also that the determinant D may be expanded according to the elements a_1, b_1, c_1 its *first row*, viz.,

$$D = a_1A_1 - b_1B_1 + c_1C_1.$$

Since any row may be interchanged with the first row if we change the sign of D, we obtain the following expansions according to the elements of the second or of the third row:

$$D = -a_2A_2 + b_2B_2 - c_2C_2, \quad D = a_3A_3 - b_3B_3 + c_3C_3.$$

For example, the determinant (27) may be expanded thus:

$$D = 0\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 2 & 1 \\ 6 & 1 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix} \equiv \begin{vmatrix} 2 & 1 \\ 6 & 2 \end{vmatrix} = -2.$$

Applying the present theorem that the rows and columns of a determinant may be interchanged to the results of §§ 39, 40, we may conclude that *the value of a determinant of the third order is changed in sign when any two of its rows are interchanged*, and also that *a determinant of the third order vanishes if any two of its rows are alike*. Similarly, from the last section we derive the theorem that *a determinant of the third order is not altered in value if we multiply the elements of any row by a constant and add the results to the elements of another row.*

47. Theorem. *A factor common to the elements of any column or any row of a determinant of the third order may be removed and placed as a factor in front of the resulting determinant.*

The proof follows from the relations:

$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} \equiv ma_1A_1 - ma_2A_2 + ma_3A_3 = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

$$\begin{vmatrix} ma_1 & mb_1 & mc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv ma_1A_1 - mb_1B_1 + mc_1C_1 = m\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and similar relations derived by interchanging rows or columns.

As an application, consider the cyclic determinant:

$$\Delta \equiv \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

Upon adding the last two columns to the first column, the resulting determinant, equal in value to Δ by § 45, has all the elements of its first column equal to $a + b + c$, which is therefore a factor of the determinant. Since $a + b + c$ and the expansion of Δ are both symmetrical in the letters a, b, c, the remaining factor is a symmetrical expression of the second degree. Hence

$$\Delta = (a + b + c)\left[m(a^2 + b^2 + c^2) + n(ab + ac + bc)\right].$$

Hence the right member must have unity as the coefficient of a^3 and zero as the coefficient of a^2b. Hence $m = 1$, $m + n = 0$. We have therefore the important factorization

$$(28) \quad \Delta = a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-ac-bc).$$

The factor of the second degree may be decomposed into two imaginary linear factors. Let ω be the quantity introduced in § 8, so that $\omega^3 = 1$. Multiplying the elements of the second column of Δ by ω and those of the third column by ω^2 and adding the products to the elements of the first column, the sums are

$$a + b\omega + c\omega^2, \quad c + a\omega + b\omega^2 \equiv \omega(a + b\omega + c\omega^2),$$
$$b + c\omega + a\omega^2 \equiv \omega^2(a + b\omega + c\omega^2).$$

Hence $a + b\omega + c\omega^2$ is a factor of Δ. Employing the multipliers ω^2 and ω, we obtain the factor $a + b\omega^2 + c\omega$. Hence

$$(28') \quad \Delta = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega).$$

EXERCISES.

Solve the following sets of simultaneous equations:

1. $x + y + 3z = 0$, $x + 2y + 2z = 0$, $x + 5y - z = 0$.

2. $x + 3y + 4z = 0$, $4x + 12y + 16z = 0$, $3x + 9y + 12z = 0$.

3. $2x - y + 4z = 0$, $x + 3y - 2z = 0$, $x - 11y + 14z = 0$.

Factor the following determinants without expanding them:

4. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ **5.** $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ **6.** $\begin{vmatrix} d & d & d \\ c & b & a \\ ab & ac & bc \end{vmatrix}$.

7. $\begin{vmatrix} a^2 & a^2-(b-c)^2 & bc \\ b^2 & b^2-(c-a)^2 & ca \\ c^2 & c^2-(a-b)^2 & ab \end{vmatrix}$ **8.** $\begin{vmatrix} a^2 & b^2 & c^2 \\ bc & ca & ab \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix}$.

48. As definition of a **determinant of the fourth order, we take***

$$(29)\qquad \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \equiv a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4,$$

where A_1 denotes the **minor** of a_1 obtained by erasing the row and the column containing a_1, A_2 the minor of a_2, etc.; that is,

$$A_1 = \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix},\quad A_2 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix},\quad A_3 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix},\quad A_4 = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

The terms of (29) which involve b_1 are seen to be

$$-a_2b_1\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} + a_3b_1\begin{vmatrix} c_2 & d_2 \\ c_4 & d_4 \end{vmatrix} - a_4b_1\begin{vmatrix} c_2 & d_2 \\ c_3 & d_3 \end{vmatrix} \equiv -b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix},$$

which equals $-b_1B_1$, where B_1 denotes the minor of b_1 in the determinant D defined by (29). Similarly, the terms of D which involve b_2 are

$$a_1b_2\begin{vmatrix} c_3 & d_3 \\ c_4 & d_4 \end{vmatrix} - a_3b_2\begin{vmatrix} c_1 & d_1 \\ c_4 & d_4 \end{vmatrix} + a_4b_2\begin{vmatrix} c_1 & d_1 \\ c_3 & d_3 \end{vmatrix} \equiv b_2\begin{vmatrix} a_1 & c_1 & d_1 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix},$$

which equals b_2B_2, where B_2 denotes the minor of b_2 in D. Similarly, the terms of D which involve b_3 may be combined into $-b_3B_3$, the terms which involve b_4 may be combined into b_4B_4. Hence we have

$$(30)\qquad D = -b_1B_1 + b_2B_2 - b_3B_3 + b_4B_4.$$

But by interchanging the first and second columns of D, we get

$$\begin{vmatrix} b_1 & a_1 & c_1 & d_1 \\ b_2 & a_2 & c_2 & d_2 \\ b_3 & a_3 & c_3 & d_3 \\ b_4 & a_4 & c_4 & d_4 \end{vmatrix} \equiv b_1B_1 - b_2B_2 + b_3B_3 - b_4B_4 = -D.$$

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* The definitions of determinants of orders 5, 6, etc., are quite similar. They have properties analogous to those of determinants of orders 2, 3, 4.

Hence a determinant of the fourth order is changed in sign by interchanging its first and second columns.

Upon interchanging the b's and c's in the identity (29), we note that the signs of A_1, A_2, A_3, A_4 are changed (§ 40). Hence the new determinant of the fourth order is the negative of D. Similarly, upon interchanging the b's and d's, or the c's and d's, the sign of D is changed.

In order to interchange the a's with the c's, we may first interchange the a's with the b's, then interchange the a's with the c's, and finally interchange the b's with the c's. The sign of the determinant is changed at each of the three steps.

The preceding results may be combined into the theorem:

A determinant of the fourth order is changed in sign by interchanging any two of its columns.

As a corollary to this theorem, we have the result:

A determinant of the fourth order vanishes if any two of its columns are alike.

49. By the definition of the determinant symbol, we have

$$\begin{vmatrix} a_1+mb_1 & b_1 & c_1 & d_1 \\ a_2+mb_2 & b_2 & c_2 & d_2 \\ a_3+mb_3 & b_3 & c_3 & d_3 \\ a_4+mb_4 & b_4 & c_4 & d_4 \end{vmatrix} = (a_1+mb_1)A_1 - (a_2+mb_2)A_2 + (a_3+mb_3)A_3 - (a_4+mb_4)A_4$$

$$= (a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4) + m(b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4).$$

The first quantity in parenthesis equals the determinant (29); the second equals m times a similar determinant with the a's replaced by b's, a determinant having its first and second columns alike and therefore equal to zero (corollary of § 48). We have therefore proved that a determinant of the fourth order is not altered in value if the elements of the second column be multiplied by a constant m and the results added to the first column.

As in § 45, we extend the theorem to the case in which the columns in question are arbitrary, say the ith and jth columns. Indeed, by an even number of interchanges of columns, we may bring the ith column into the position of the first column, and the jth into the second column. Each interchange of columns only changes the sign of the determinant (§ 48). We may therefore state the theorem:

A determinant of the fourth order is not altered in value if the elements of any column are multiplied by a constant and the products added to the corresponding elements of any other column.

As an application, we prove that the determinant

$$\Delta \equiv \begin{vmatrix} a_1 & nc_1 & c_1 & d_1 \\ a_2 & nc_2 & c_2 & d_2 \\ a_3 & nc_3 & c_3 & d_3 \\ a_4 & nc_4 & c_4 & d_4 \end{vmatrix}$$

vanishes identically. Multiplying the third column by $-n$ and adding to the second column, we obtain an equal determinant having only zeros in the second column. Interchanging the first and second columns, we obtain a determinant which vanishes, as shown by the definition (29), whereas its value is $-\Delta$.

50. Theorem. *A determinant of the fourth order is unaltered in value if its corresponding rows and columns are interchanged.*

In fact, if the given determinant is (29), the resulting determinant is

$$D \equiv \begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{vmatrix} = a_1A_1 - b_1B_1 + c_1C_1 - d_1D_1,$$

where A_1 is the minor of a_1, B_1 the minor of b_1, etc., in D. In view of the corresponding theorem for determinants of the third order (§ 46), these determinants A_1, B_1, C_1, D_1 are equal to the minors of a_1, b_1, c_1, d_1, respectively, in the original determinant (29). Hence in the two determinants D and (29), the terms involving a_1 are the same. The terms of (29) which involve b_1 were seen in § 48 to be given by $-b_1B_1$, the same as in D. The terms of (29) which involve c_1 are evidently

$$(-a_2)(-c_1)\begin{vmatrix} b_3 & d_3 \\ b_4 & d_4 \end{vmatrix} + a_3(-c_1)\begin{vmatrix} b_2 & d_2 \\ b_4 & d_4 \end{vmatrix} - a_4(-c_1)\begin{vmatrix} b_2 & d_2 \\ b_3 & d_3 \end{vmatrix} = c_1C_1.$$

Similarly, the terms of (29) which involve d_1 are given by $-d_1D_1$.

It follows that a determinant (29) of the fourth order can be expanded according to the elements of the first *row* by taking the sum of the products of each element and its minor, with the proper sign prefixed, viz., $a_1A_1 - b_1B_1 + c_1C_1 - d_1D_1$.

51. Combining the theorem of § 50 with the theorems of §§ 48, 49, we derive at once the following results:

A determinant of the fourth order is changed in sign by interchanging any two of its rows. It vanishes if any two of its rows are alike. It is not altered in value if the elements of any row are multiplied by a constant and the products added to the corresponding elements of any other row.

52. By the definition (29) and the proof in § 50, a determinant D of the fourth order can be expanded according to the elements of the first column or of the first row as follows:

$$D = a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4, \qquad D = a_1A_1 - b_1B_1 + c_1C_1 - d_1D_1.$$

Since any row can be brought into the first row by an interchange of rows, and likewise any column into the first column, the determinant being changed in sign by each interchange, we derive the expansions:

$$D = -b_1B_1 + b_2B_2 - b_3B_3 + b_4B_4, \qquad D = -a_2A_2 + b_2B_2 - c_2C_2 + d_2D_2$$
$$D = +c_1C_1 - c_2C_2 + c_3C_3 - c_4C_4, \qquad D = +a_3A_3 - b_3B_3 + c_3C_3 - d_3D_3,$$
$$D = -d_1D_1 + d_2D_2 - d_3D_3 + d_4D_4, \qquad D = -a_4A_4 + b_4B_4 - c_4C_4 + d_4D_4.$$

The signs are determined by the positions of the elements. The following scheme, analogous to a checker-board, is useful:

+	−	+	−
−	+	−	+
+	−	+	−
−	+	−	+

53. As an example, we evaluate the determinant

$$D \equiv \begin{vmatrix} 2 & -1 & 3 & -2 \\ 1 & 7 & 1 & -1 \\ 3 & 5 & -5 & 3 \\ 4 & -3 & 2 & -1 \end{vmatrix}.$$

We seek an equal determinant having three zeros in one row or one column. To avoid fractions, we select a row or column containing ± 1 as an element, say the last column.* Adding it to the first and third columns, and, after multiplication by 7, to the second column, we get

$$D = \begin{vmatrix} 0 & -15 & 1 & -2 \\ 0 & 0 & 0 & -1 \\ 6 & 26 & -2 & 3 \\ 3 & -10 & 1 & -1 \end{vmatrix} = -\begin{vmatrix} 0 & -15 & 1 \\ 6 & 26 & -2 \\ 3 & -10 & 1 \end{vmatrix},$$

upon expansion according to the second row. Multiplying the last row of the determinant of the third order by -2 and adding to the second row, we get

$$D = -\begin{vmatrix} 0 & -15 & 1 \\ 0 & 46 & -4 \\ 3 & -10 & 1 \end{vmatrix} = -3\begin{vmatrix} -15 & 1 \\ 46 & -4 \end{vmatrix} = -42.$$

54. Consider the general set of simultaneous equations

$$\begin{aligned} a_1x + b_1y + c_1z + d_1w &= k_1, \\ a_2x + b_2y + c_2z + d_2w &= k_2, \\ a_3x + b_3y + c_3z + d_3w &= k_3, \\ a_4x + b_4y + c_4z + d_4w &= k_4. \end{aligned}$$

To solve for x, multiply the first equation by the minor A_1, the second equation by $-A_2$, the third by A_3, and the fourth by $-A_4$, and add the resulting equations. By formula (29), the coefficient of x in the sum is the determinant D of the coefficients of the left members. The resulting equation becomes

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* If none were present, we first work to that end, unless all the elements of one row or column are multiples of one element (as is the case with the first column of the above determinant of the third order).

$$Dx+(b_1A_1-b_2A_2+b_3A_3-b_4A_4)y+(c_1A_1-c_2A_2+c_3A_3-c_4A_4)z$$
$$+(d_1A_1-d_2A_2+d_3A_3-d_4A_4)w = k_1A_1-k_2A_2+k_3A_3-k_4A_4.$$

The coefficients of y, z, w all vanish since they are the expansions of determinants having two columns alike (§ 48). The right member is the expansion of a determinant derived from D upon replacing the a's by k's. Hence the relation becomes

$$Dx \equiv \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} x = \begin{vmatrix} k_1 & b_1 & c_1 & d_1 \\ k_2 & b_2 & c_2 & d_2 \\ k_3 & b_3 & c_3 & d_3 \\ k_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

To solve for y, we use the multipliers $-B_1$, $+B_2$, $-B_3$, $+B_4$. The coefficients of x, z, w are seen to vanish, while that of y is, by (30),

$$-b_1B_1+b_2B_2-b_3B_3+b_4B_4 = D.$$

To solve for z, we use the multipliers $+C_1$, $-C_2$, $+C_3$, $-C_4$. Similarly for w. We obtain the following results:

$$Dy = \begin{vmatrix} a_1 & k_1 & c_1 & d_1 \\ a_2 & k_2 & c_2 & d_2 \\ a_3 & k_3 & c_3 & d_3 \\ a_4 & k_4 & c_4 & d_4 \end{vmatrix}, \quad Dz = \begin{vmatrix} a_1 & c_1 & k_1 & d_1 \\ a_2 & c_2 & k_2 & d_2 \\ a_3 & c_3 & k_3 & d_3 \\ a_4 & c_4 & k_4 & d_4 \end{vmatrix}, \quad Dw = \begin{vmatrix} a_1 & b_1 & c_1 & k_1 \\ a_2 & b_2 & c_2 & k_2 \\ a_3 & b_3 & c_3 & k_3 \\ a_4 & b_4 & c_4 & k_4 \end{vmatrix}.$$

Hence the product of any one of the unknown quantities by the determinant D equals the determinant obtained from D upon replacing the coefficients of that unknown by the constants $k_1, \ldots, k_4$.

As an example we take the simultaneous equations

$$\begin{aligned} x+y+z+w &= 0, \\ x+2y+3z+4w &= 0, \\ x+3y+6z+10w &= 0, \\ x+4y+10z+20w &= -1. \end{aligned}$$

The determinant D of the coefficients of the left members is found to equal $+1$. Hence

$$x = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \\ 0 & 3 & 6 & 10 \\ -1 & 4 & 10 & 20 \end{vmatrix} = +1 \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 6 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 3 & 7 \end{vmatrix} = 1,$$

$$y = \begin{vmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 3 & 4 \\ 1 & 0 & 6 & 10 \\ 1 & -1 & 10 & 20 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 4 \\ 1 & 6 & 10 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 0 & 5 & 9 \end{vmatrix} = -\begin{vmatrix} 2 & 3 \\ 5 & 9 \end{vmatrix} = -3.$$

Similarly, $z = 3$, $w = -1$. These values of x, y, z, w satisfy the given equations.

EXERCISES.

Solve by determinants the sets of simultaneous equations:

1. $$\begin{aligned} 2x - y + 3z - 2w &= 14, \\ x + 7y + z - w &= 13, \\ 3x + 5y - 5z + 3w &= 11, \\ 4x - 3y + 2z - w &= 21. \end{aligned}$$

2. $$\begin{aligned} ax + by + cz + dw &= k, \\ a^2x + b^2y + c^2z + d^2w &= k^2, \\ a^3x + b^3y + c^3z + d^3w &= k^3, \\ a^4x + b^4y + c^4z + d^4w &= k^4. \end{aligned}$$

3. $$\begin{aligned} 6x + 4y + 3z - 84w &= 0, \\ x + 2y + 3z - 48w &= 0, \\ x - 2y + z - 12w &= 0, \\ 4x + 4y - z - 24w &= 0. \end{aligned}$$

4. $$\begin{aligned} 2x + 3y - 4z + 5w &= 0, \\ 3x + 5y - z + 2w &= 0, \\ 7x + 11y - 9z + 12w &= 0, \\ 3x + 4y - 11z + 13w &= 0. \end{aligned}$$

Factor the determinants without using their expansions:

5. $$\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}.$$

6. $$\begin{vmatrix} 0 & a & b & c \\ a & 0 & c & b \\ b & c & 0 & a \\ c & b & a & 0 \end{vmatrix}.$$

Solve by determinants the simultaneous equations in three unknown quantities to which the following problems lead:

7. The value of 370 coins consisting of dollars, dimes, and half-dimes amounts to \$61 If there were twice as many dimes, half as many half-dimes, and three times as many dollars, the total value would be \$144. Find the number of coins of each kind.

8. How much money shall be given to A, B, C so that A shall receive one-ninth as much as B and C together, B one-third as much as A and C together, and C \$6 more than A and B together?

9. Find a number such that the sum of its three digits is 16, the first digit equals the sum of the second and third digits, the sum of the first and second digits equals seven times the third digit.

10. Find a number the sum of whose three digits is 15, which is increased by 99 upon reversing its digits, and which is decreased by 9 upon interchanging the second and third digits.

On the history of determinants, see Muir, Theory of Determinants in the Historical Order of its Development, London (Macmillan & Co.), 1890; Pascal, Die Determinanten, Leipzig (Teubner), 1900.

In addition to the latter, the following text-books may be consulted:

Scott, Theory of Determinants, Cambridge.
Muir, Treatise on the Theory of Determinants (Macmillan).
Weld, The Theory of Determinants (Macmillan), 1893.
Baltzer, Theorie und Anwendung der Determinanten, Leipzig.
Burnside and Panton, Theory of Equations, Dublin (pp. 229-288).
Merriman and Woodward's Higher Mathematics, New York (Wiley & Sons).

CHAPTER V.

RATIO; PROPORTION; VARIATION.

55. By the **ratio of two algebraic numbers** a and b is meant the quantitative relation of a to b which is measured by the quotient $\frac{a}{b}$. A notation for this ratio is $a : b$. By the ratio of any two quantities of the same kind is meant the ratio of the two numbers expressing the number of units contained in the quantities. Thus the ratio of 2 months to 10 days is expressed by the number $\frac{60}{10} = 6$; the ratio of \$2 to 20 cents is expressed by the number 10.

Evidently the ratio $a : b$ equals the ratio $ma : mb$. The ratio of $\frac{c}{d}$ to $\frac{r}{s}$ equals $\frac{c}{d} \div \frac{r}{s} = \frac{cs}{dr}$.

If the ratio of two quantities can be expressed as a rational number, the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable.** Thus the diagonal and side of a square are incommensurable, since their ratio $= \frac{\sqrt{2}}{1} \equiv \sqrt{2}$ is not a rational number (§ 2).

The ratio $a\alpha : b\beta$ is said to be the ratio **compounded** of the ratios $a : b$ and $\alpha : \beta$. Its value $\frac{a\alpha}{b\beta}$ equals the product of $\frac{a}{b}$ and $\frac{\alpha}{\beta}$. When the ratio $a : b$ is compounded with itself the resulting ratio $a^2 : b^2$ is called the **duplicate ratio** of $a : b$. Likewise $a^3 : b^3$ is called the **triplicate ratio** of $a : b$.

56. Problems involving several equal ratios

$$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots = r \tag{1}$$

are usually solved most simply by eliminating the numerators a, c, e, . . . of the fractions (1), by substituting in the proposed identity their values

$$a = rb, \quad c = rd, \quad e = rf, \quad \ldots . \tag{2}$$

EXAMPLE 1. Given $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that

$$\frac{a^3 - 2c^3 + 4e^3}{b^3 - 2d^3 + 4f^3} = \frac{ace}{bdf}.$$

Making the substitution (2), the two fractions become

$$\frac{r^3b^3 - 2r^3d^3 + 4r^3f^3}{b^3 - 2d^3 + 4f^3} = r^3, \quad \frac{r^3bdf}{bdf} = r^3.$$

EXAMPLE 2. If the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}, \ldots$ are equal, each equals

$$\frac{a + c + e + \ldots}{b + d + f + \ldots}.$$

Making the substitution (2), the fraction reduces to r.

EXAMPLE 3. Prove that $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$, when it is given that

$$\frac{bz - cy}{a} = \frac{cx - az}{b} = \frac{ay - bx}{c}.$$

Applying the result of Ex. 2, each of the latter fractions equals

$$\frac{a(bz - cy) + b(cx - az) + c(ay - bx)}{a^2 + b^2 + c^2} \equiv 0.$$

Hence $bz - cy = 0$, $cx - az = 0$, $ay - bx = 0$, and therefore

$$\frac{z}{c} = \frac{y}{b}, \quad \frac{x}{a} = \frac{z}{c}, \quad \frac{y}{b} = \frac{x}{a}.$$

EXERCISES.

1. Arrange the ratios 5 : 6, 14 : 16, 41 : 48, 31 : 36 in descending order of magnitude. Find the ratio compounded of them.

2 What value of x makes $3 + x : 4 + x = 5 : 6$?

3. Find the duplicate and triplicate ratios of 3 : 5, also of 6 : 7.

4. Find two numbers such that their difference, their sum, and the sum of their squares are as 1 : 15 : 113.

5. Find two numbers whose sum is 77 and whose ratio is 3 : 8.

If $a : b = c : d = e : f$, prove that

6. $a^4b^2 + 3a^2e^2 - 5e^4f : b^6 + 3b^2f^2 - 5f^5 = a^4 : b^4$.

7. $a^2c + 3c^3e + 7e^2c : b^2d + 3d^3e + 7f^2d = e^3 : f^3$.

8. $\sqrt{(ac - 3c^2 + 4e^2)} : \sqrt{(bd - 3d^2 + 4f^2)} = c : d$.

9. $a^2 + ab : c^2 + cd = b^2 - 5ab : d^2 - 5cd$.

10. If $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$, then $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$.

11. If $l(my + nz - lx) = m(nz + lx - my) = n(lx + my - nz)$, prove that $y + z - x : l = z + x - y : m = x + y - z : n$.

12. If $\frac{4x - 3y}{3z + y} = \frac{z - y}{z - x} = \frac{2x + 3z}{5y - 3x}$, each fraction equals $\frac{x}{y}$.

13. If $\frac{x}{a - b} = \frac{y}{b - c} = \frac{z}{c - a}$, then $x + y + z = 0$.

57. Proportion. If the ratio $a : b$ equals the ratio $c : d$, the four quantities a, b, c, d are said to be **proportionals** or **in proportion.** The proportion is written $a : b :: c : d$, or also $a : b = c : d$, and is read a is to b as c is to d. The terms a and d are called the **extremes,** b and c the **means.**

In any proportion the product of the extremes equals the product of the means. Thus, from $a : b = c : d$ we get

$$\frac{a}{b} = \frac{c}{d}, \quad ad = bc.$$

Inversely, if $ad = bc$, then a, b, c, d are in proportion.

If three quantities a, b, c are such that $a : b = b : c$, so that $b^2 = ac$, then b is said to be a **mean proportional** between a and c, while c is said to be a **third proportional** to a and b.

If $a : b = c : d$, then $a \pm b : b = c \pm d : d$. In proof, we add ± 1 to each term of $\frac{a}{b} = \frac{c}{d}$. Then $\frac{a \pm b}{b} = \frac{c \pm d}{d}$.

Dividing the terms of the equation in which the plus sign holds by the terms of the equation in which the minus sign holds, we get

$$\frac{a + b}{a - b} = \frac{c + d}{c - d}.$$

Hence, if $a : b = c : d$, then $a + b : a - b = c + d : c - d$, a result said to be derived **by composition and division** from $a : b = c : d$.

EXERCISES.

1. If $a : b = b : c$, then $a : c = a^2 : b^2$.
2. If $a : b = c : d$, then $b : a = d : c$, and $a : c = b : d$.
3. If $a : b = c : d$, and $e : f = g : h$, then $ae : bf = cg : dh$.
4. Find the mean and third proportionals to 4 and 16; to a^4 and a^2b^2.

If $a : b = c : d$, prove that

5. $a^2c - ac^2 : b^2d - bd^2 = (a - c)^3 : (b - d)^3$.
6. $pa^2 + qb^2 : pa^2 - qb^2 = pc^2 + qd^2 : pc^2 - qd^2$.
7. $b - d : a - c = \sqrt{b^2 + d^2} : \sqrt{a^2 + c^2}$.

If $a : b = b : c = c : d$, prove that

8. $a : b + d = c^3 : c^2d + d^3$.
9. $a + 5d : 2a - 3d = a^3 + 5b^3 : 2a^3 - 3b^3$.
10. $(a^2 + b^2 + c^2)(b^2 + c^2 + d^2) = (ab + bc + cd)^2$.
11. Find four proportionals the sum of whose squares is 530, the sum of the extremes being 23 and the sum of the means 13.

58. Variation. If two variable quantities A and B depend upon each other in such a way that when A is changed in a certain ratio B is changed in the same ratio, B is said to **vary directly** as A. For example, a body moving at the uniform rate of 5 miles an hour will move 10 miles in 2 hours, 30 miles in 6 hours, etc., so that the distance varies as the time. Between the distance B reckoned in miles and the time A reckoned in hours, the following relation holds: $B = 5A$. In the general case, if B varies directly as A, then $B = mA$, where m is some constant number.

If B varies directly as the reciprocal (or inverse) of A, then B is said to **vary inversely** as A, and $B = \frac{c}{A}$, where c is some constant number.

59. The Law of Boyle states that the volume of a given mass of any gas at a constant temperature varies inversely as the pressure. Thus if V is the volume when the pressure is P, the volume becomes $\frac{1}{3}V$ when the pressure is $3P$, the volume becomes nV when the pressure is $\frac{1}{n}P$.

The Law of Gay-Lussac and Charles states that, if the pressure be constant, the volume of a mass of gas varies directly as the temperature measured in degrees Centigrade* above $-273°$ C. A gas therefore expands by $\frac{1}{273}$ of its volume at 0° for every increase in temperature of 1° Centigrade. Thus 273 volumes of air at 0° become $273 + t$ volumes at $t°$ Centigrade, and the latter become $273 + T$ volumes at $T°$.

From the preceding law and the Law of Boyle, we would expect that, when the pressure P and the temperature T (in degrees above $-273°$ C.) both vary, the volume V of the gas would vary as $\frac{T}{P}$. By actual experiment this statement is found to be true for moderate values of T and P. It may however be derived from the two laws stated by means of the following algebraic theorem:

60. *If A depends only on B and C, and if A varies as B when C is constant, and if A varies as C when B is constant, then A will vary as the product BC when B and C change simultaneously.*

Let the quantities A, B, C have initially the values a, b, c and suppose that a change in B from b to β and a *simultaneous* change in C from c to γ together cause a change in A from a to α. We may, however, consider that the changes in B and C take place *successively*. First, if C remains at the constant value c while B changes from b to β, then A will change from a to some value a' intermediate to a and α. Applying the hypothesis for this case, we get

$$\frac{a}{a'} = \frac{b}{\beta}.$$

Next, let B remain at the constant value β, which it just attained, while C now changes from c to γ. Then A must complete its change from the intermediate value a' to the final value α. Applying the hypothesis for such a change, we get

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* The freezing-point of pure water is 32° Fahrenheit or 0° Centigrade; the boiling-point of water is 212° Fahrenheit or 100° Centigrade. Hence 180 degrees Fahr. = 100 degrees Cent.

$$\frac{a'}{\alpha} = \frac{c}{\gamma}.$$

Multiplying together the two equations, we get

$$\frac{a}{\alpha} = \frac{bc}{\beta\gamma}.$$

Hence, if B and C change simultaneously, A varies as BC.

EXAMPLE.—Triangles with the same base are proportional to their altitudes, and triangles with the same altitude are proportional to their bases. Hence the areas of triangles vary as the products of their bases and altitudes.

EXERCISES.

1. If x varies as y and if $x = 6$ when $y = 15$, find x when $y = 10$.

2. If x and y each vary as z, then $x + y$ and $\sqrt{xy}$ each vary as z.

3. If y varies as $a + b$, and a varies directly as x, and b varies inversely as x^2, and if $y = 19$ when $x = 2$ or 3, find y in terms of x.

4. If x varies directly as y and inversely as z, and if $x = 2$ when $y = 3$ and $z = 4$, find y when $x = 12$ and $z = 6$.

5. When a body falls from rest its distance from the initial point varies as the square of the time it has been falling, and its velocity varies as the time. If a body falls 400 feet from rest in 5 seconds, how far does it fall in 10 seconds? How far in the tenth second? If the velocity at the end of two seconds is 64, what is the velocity at the end of 5 seconds? At the end of 10 seconds?

6. An amount of gas measures 100 cubic feet at 0°; find its volume at 10° Centigrade, the pressure remaining constant.

7. Given 500 cubic feet of air at 10° Centigrade, find its volume at – 10° C., the pressure being unchanged.

8. If the temperature of a gas is raised from 0° to 30° Centigrade, and the pressure increased tenfold, what becomes of 500 cu. ft. of air?

9. What is the increase in pressure of the air in an air-tight room, when the temperature is raised from 0° to 40° Centigrade?

10. Given that the area of a circle varies as the square of its radius, show that a circle of 5 inches radius equals the sum of two circles of radii 3 and 4 inches.

CHAPTER VI.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSIONS.

61. A series of quantities is said to be in **arithmetical progression** when the difference between any term (after the first) and the preceding term is the same throughout the series.

The following series are examples of arithmetical progressions:

$$\begin{array}{cccccc} 2, & 4, & 6, & 8, \ldots, & 2n, \ldots \\ -1, & -3, & -5, & -7, \ldots, & -(2n-1), \ldots \\ a, & a+d, & a+2d, & a+3d, \ldots, & a+(n-1)d, \ldots \end{array}$$

In the first series the **common difference** is 2, in the second series it is -2, in the third series it is d. The nth term in the first series is $2n$, in the second series $-(2n-1)$, in the third series $a+(n-1)d$. In particular, for $n=4$, the fourth terms in the respective series are $2\cdot 4=8$, $-(2\cdot 4-1)=-7$, $a+(4-1)d=a+3d$. We may state the theorem:

In an arithmetical progression with the first term a and the common difference d, the nth term is $a+(n-1)d$.

62. When three quantities are in arithmetical progression, the middle one is called the **arithmetical mean** of the other two.

But, if a, b, c are in A. P.*, we have

$$b-a=c-b=\text{common difference.}$$

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* A. P. is an abbreviation for arithmetical progression. Similarly, G. P. will be used as an abbreviation for geometrical progression, H. P. for harmonical progression.

Hence $b = \frac{1}{2}(a + c)$, so that *the arithmetical mean of any two quantities is half their sum.* It is their "average value."

When any number of quantities are in arithmetical progression, the terms between the first and last are called **arithmetical means** between the first and last quantities.

Thus 7, 10, 13, 16, 19 are in A. P. and 10, 13, 16 are the three arithmetical means between 7 and 19.

63. *Sum of any number of terms in arithmetical progression.*

Let S denote the sum of n terms in A.P. of which a is the first term, l the last (viz., the nth) term, and d the common difference. By § 61, $l = a + (n - 1)d$. We have

$$S = a + (a + d) + (a + 2d) + \dots + (l - 2d) + (l - d) + l.$$

Writing the series in reverse order, we get

$$S = l + (l - d) + (l - 2d) + \dots + (a + 2d) + (a + d) + a.$$

Adding the corresponding terms of the two series, we get

$$2S = (a + l) + (a + l) + (a + l) + \dots + (a + l) = n(a + l),$$

there being n terms each $(a + l)$. Hence

$$(1) \qquad S = \tfrac{1}{2}n(a + l).$$

Here $\frac{1}{2}(a + l)$ is the average of the first and last terms of the series, also the average of the second term and the term preceding the last, etc. If n is odd, there is a middle term, the $\frac{1}{2}(n + 1)$st term of the series, whose value is seen to be

$$a + \{\tfrac{1}{2}(n + 1) - 1\}d = \tfrac{1}{2}\{a + a + (n - 1)d\} = \tfrac{1}{2}(a + l).$$

Hence $\frac{1}{2}(a + l)$ is the average throughout the series. We derive a second proof that the sum of n terms is $n\dfrac{a + l}{2}$.

64. EXAMPLE 1. Find the sum of 19 terms of the A. P. 1, 5, 9, . . .
The 19th term l is $1 + (19 - 1)4 = 73$. The required sum is

$$19 \cdot \frac{1 + 73}{2} = 19 \cdot 37 = 703.$$

EXAMPLE 2. The 11th term of an A. P. is 12, and the 19th term is 36; find the 40th term.

The conditions give

$$a + 10d = 12, \quad a + 18d = 36.$$

Hence $8d = 24$, $d = 3$, whence $a = -18$. Then the 40th term is

$$a + 39d = -18 + 39 \cdot 3 = 99.$$

EXAMPLE 3. Insert six arithmetical means between 8 and 29.

We are to construct an A.P. of 8 terms such that the first term a is 8 and the eighth term $a + 7d$ is 29. Hence $7d = 21$, $d = 3$. The required means are therefore $a + d = 11$, $a + 2d = 14$, 17, 20, 23, 26.

EXAMPLE 4. How many terms of the A. P. 48, 40, 32, . . . must be taken so that the sum may be 144?

Let the number of terms be n. Since the common difference is -8, the nth term is $l \equiv 48 - 8(n-1)$. Hence

$$144 = S = \tfrac{1}{2}n(a + l) = \tfrac{1}{2}n\{96 - 8(n-1)\}.$$

Hence $144 = n(52 - 4n)$, so that

$$n^2 - 13n + 36 \equiv (n-4)(n-9) = 0.$$

For $n = 9$, we get the A. P. 48, 40, 32, 24, 16, 8, 0, -8, -16, whose sum is 144. Since the last five terms have the sum zero, $n = 4$ is a suitable value.

EXERCISES.

1. Find the 13th and 41st terms of the series 6, 12, 18, . . .
2. Find the 20th and 40th terms of the series -5, -3, -1, . . .
3. Find the 10th and 60th terms of the series 1, 6, 11, . . .

Find the sum of the following series:

4. 5, 11, 17, . . . to 30 terms.
5. 12, 9, 6, . . . to 21 terms.
6. $2\frac{1}{2}$, 4, $5\frac{1}{2}$, . . . to 37 terms.
7. a, $3a$, $5a$, . . . to a terms.
8. a, 0, $-a$, . . . to $4a$ terms.
9. 13, 9, 5, . . . to 100 terms.

Find the common difference and the number of terms in the A. P.:

10. The first term is 6, the last term 180, the sum 2790.
11. The sum is 72, the first term 27, the last term -18.
12. The last term is -32, the sum -266, the first term -6.
13. The first term is a, the last term $13a$, the sum $49a$.

Find the A.P. in which

14. The 7th term is 1 and the 31st term is -77.
15. The 12th term is 214 and the 41st term is 739.
16. The 54th term is -125 and the 4th term is zero.

How many terms must be taken of

17. The series 15, 12, 9, . . . to make the sum 45?
18. The series 42, 39, 36, . . . to make the sum 315?

19. The series 16, 15, 14, . . . to make the sum 100?

20. The series $-10\frac{1}{3}$, -9, $-7\frac{2}{3}$ to make the sum -42?

21. Insert 7 arithmetical means between 269 and 295.

22. Insert 15 arithmetical means between 23 and 71.

23. Insert 8 arithmetical means between -80 and -50.

24. Insert 10 arithmetical means between $6x - 5y$ and $6y - 5x$.

25. Find 5 numbers in A. P. whose sum is 80 and the sum of whose squares is 1640.

26. Find three numbers in A.P. whose sum is 39 and product 2184.

27. The sum of the first n odd numbers is n^2.

28. Find the sum of all the odd numbers between 200 and 400.

65. A series of quantities is said to be in **geometrical progression** when the ratio of any term (after the first) to the preceding term is the same throughout the series.

The following series are examples of geometrical progressions:

$$\begin{array}{llllllll}
1, & 2, & 4, & 8, & \ldots, & 2^{n-1}, & 2^n, & \ldots \\
9, & 3, & 1, & \frac{1}{3}, & \ldots, & (\frac{1}{3})^{n-3}, & (\frac{1}{3})^{n-2}, & \ldots \\
a, & ar, & ar^2, & ar^3, & \ldots, & ar^{n-1}, & ar^n, & \ldots
\end{array}$$

In the first series the **common ratio** is 2, in the second series it is $\frac{1}{3}$, in the third series it is r. The nth term in the first series is 2^{n-1}, in the second series $9(\frac{1}{3})^{n-1}$, in the third series ar^{n-1}. In particular, for $n = 4$, the fourth terms in the respective series are $2^3 = 8$, $9(\frac{1}{3})^3 = \frac{1}{3}$, ar^3. We may state the theorem:

In a geometrical progression with the first term a and the common ratio r, the nth term is ar^{n-1}.

66. When three quantities are in geometrical progression, the middle one is called the **geometrical mean** of the other two.

But, if a, b, c are in G.P., we have

$$\frac{b}{a} = \frac{c}{b} = \text{common ratio.}$$

Hence $b^2 = ac$, so that *the geometrical mean of any two quantities is the square root of their product.* It follows from § 57 that the geometrical mean of a and c is the mean proportional between a and c.

When any number of quantities are in geometrical progression, the terms between the first and last are called **geometrical means** between the first and last quantities.

Thus 2, 10, 50, 250, 1250 are in G. P., and 10, 50, 250 are the three geometrical means between 2 and 1250.

67. Theorem. *The sum of n terms of a geometrical progression whose first term is a and common ratio r is*

$$S_n \equiv \frac{a(1-r^n)}{1-r}. \tag{2}$$

Since the nth term is ar^{n-1}, we have

$$S_n = a + ar + ar^2 + \ldots + ar^{n-1} = a(1 + r + r^2 + \ldots + r^{n-1}).$$

But formula (1) of § 28 gives

$$1 + r + r^2 + \ldots + r^{n-1} = \frac{r^n - 1}{r - 1} = \frac{1 - r^n}{1 - r}.$$

68. As an example, the sum of 20 terms of the geometrical progression 1, $\frac{1}{2}$, $\frac{1}{4}$, . . . is

$$S_{20} = \frac{1 - (\frac{1}{2})^{20}}{1 - \frac{1}{2}} = 2 - (\tfrac{1}{2})^{19}.$$

The sum of 50 terms is $2 - (\frac{1}{2})^{49}$, the sum of 100 terms is $2 - (\frac{1}{2})^{99}$. It appears that the sum of n terms S_n is less than 2, but differs from 2 by an amount which decreases as n increases. In fact, S_{20}, S_{50}, S_{100} differ from 2 by $(\frac{1}{2})^{19}$, $(\frac{1}{2})^{49}$, $(\frac{1}{2})^{99}$, respectively. By taking n sufficiently large, S_n will differ from 2 by as little as we please. In fact, for each increase of n by unity, the difference in question is divided by 2. The statement is made more evident by observing that $(\frac{1}{2})^4 = \frac{1}{16} < \frac{1}{10}$, so that

$$\left(\frac{1}{2}\right)^{4m} < \frac{1}{10^m} \equiv .\,00\ldots01 \ (m - 1 \text{ zeros before } 1).$$

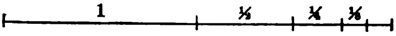
Hence $(\frac{1}{2})^{4m}$ may be made as small as we please by taking m large enough. We have established the following theorem:

By taking a sufficiently large number of terms, the sum

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{8} + \dots \tag{3}$$

can be made to differ from 2 by as little as we please.

The theorem may be illustrated by the diagram:



A line two inches long is divided into two equal parts; the second part is subdivided into two parts each ½ inch long; the second subpart is divided into two parts each ¼ inch long, etc. Taking the first of each pair of parts and forming the sum, we get the expression (3). After any number of steps, the sum is less than the length 2 inches of the whole line; but the difference from 2 inches, being the last piece of the line, tends towards zero as the operation is continued.

69. For the general series $a, ar^2, ar^3, \dots$, the sum S_n differs from $\frac{a}{1-r}$ by the quantity $\frac{ar^n}{1-r}$. If r is a proper fraction, whether positive or negative, the numerical value of r^n decreases as n increases and may be made to approach zero as near as we please by sufficiently increasing the value of n. We will denote by S_∞ the limit of the sum S_n of n terms when n increases without bound, and speak of the result as the **sum to infinity.** Hence, *in a geometrical progression with the first term a and common ratio r numerically* <1, *the sum to infinity is*

$$S_\infty = \frac{a}{1-r}. \tag{4}$$

Notice that, in a geometrical progression with $r > 1$, the result (4) no longer holds. For example, if $a = 1$, $r = 2$, the sum

$$1 + 2 + 4 + 8 + 16 + \dots + 2^{n-1}$$

increases without bound when the number of terms n increases, whereas $\frac{a}{1-r}$ has the value -1.

Example 1. Find the geometrical progression whose second term is -6 and whose sum to infinity is 8.

Let a be the first term and r the common ratio. Then

$$ar = -6, \quad \frac{a}{1-r} = 8.$$

By division, $$r(1-r) = -\frac{6}{8}.$$

Solving this quadratic equation, $r = -\frac{1}{2}$ or $+\frac{3}{2}$. The value $r = \frac{3}{2}$ is impossible, since S_∞ would then exceed 8. Hence $r = -\frac{1}{2}$, so that the series is 12, -6, 3, . . .

Example 2. Find the value of the recurring decimal

$$.6\,\dot{2}\dot{3} \equiv .6\ 23\ 23\ 23\ldots$$

Aside from .6, it may be regarded as a G. P. with the ratio $(\frac{1}{10})^2$:

$$\frac{6}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \cdot\cdot$$

$$= \frac{6}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \ldots\right)$$

$$= \frac{6}{10} + \frac{23}{10^3}\left(\frac{1}{1-\frac{1}{10^2}}\right) = \frac{6}{10} + \frac{23}{10^3} \cdot \frac{100}{99}$$

$$= \frac{6}{10} + \frac{23}{990} = \frac{617}{990}.$$

EXERCISES.

1 Sum $12 + 18 + 27 + \ldots$ to 8 terms, also to n terms.

2. Sum $36 - 12 + 4 - \ldots$ to 7 terms, also to n terms.

3. Sum $3 - 2 + \frac{4}{3} - \ldots$ to n terms also to infinity.

4. Sum $.5 + .15 + .045 + \ldots$ to n terms, also to infinity.

5. Evaluate $.02\dot{3}$; $1.4\dot{6}\dot{6}$; $.\dot{1}4285\dot{7}$; $.0\dot{5}\dot{2}$.

6. Find the G. P. with 2d term -4 and sum to infinity 9.

7. Find three numbers in G. P. such that their sum is 62, and the sum of their squares is 2604

8. If an odd number of quantities are in G. P., show that the first, the middle, and the last of them are also in G. P.

9. Insert between 4 and 18 two numbers such that the first three shall be in A. P. and the last three in G. P.

10. Find three numbers in G. P. whose product is 1728 and the sum of whose products by twos is 624.

11. Insert four geometrical means between 5 and 160; between $\frac{1}{9}$ and 27.

12. Find a G. P. with 81 as the 5th term and 24 as the 2d term.

13. Find a G. P. the sum of whose first and second terms is 5 and such that every term is 3 times the sum to infinity of all the terms that follow it.

14. Sum to n terms $x + a$, $x^2 + 2a$, $x^3 + 3a$, ...

15. Sum to $2n$ terms $a + 3$, $3a - 6$, $5a + 12$, ...

16. Find the ratio of two numbers whose arithmetical mean is double the geometrical mean.

17. Find a, b, c, given that their sum is 70, that a, b, c are in G. P., and that $4a, 5b, 4c$ are in A. P.

18. Explain the paradox of the race between the hare and the tortoise. The latter travels at the rate of 1 mile an hour and the former travels 2 miles an hour. The tortoise has a start of one hour. While the hare is covering the distance the tortoise travelled during the preceding period, the tortoise moves ahead a new distance, etc. Does the hare overtake the tortoise?

70. Several quantities are said to be in **harmonical progression** when their reciprocals are in arithmetical progression. Of three quantities in H. P., the middle one is called the **harmonical mean** of the other two.

If a, b, c are in H. P., then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A. P., whence

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}; \quad \frac{2}{b} = \frac{1}{a} + \frac{1}{c}.$$

Hence the harmonical mean b of a and c is

$$b = \frac{2ac}{a + c}.$$

To insert n **harmonical means** between a and c, we construct the H. P. of $n + 2$ terms of which a is the first and c the last term. For example, to insert 4 harmonical means between 1 and $\frac{1}{6}$, we observe that 1, 2, 3, 4, 5, 6 are in A. P. and therefore that 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$ are in H. P.

EXERCISES.

1. The geometrical mean of any two quantities is also the geometrical mean between their arithmetical and harmonical means.

2. The conditions that a, b, c may be in A. P., G. P., or H. P. are

$$a - b : b - c :: a : a,$$
$$a - b : b - c :: a : b,$$
$$a - b : b - c :: a : c, \text{ respectively.}$$

3. If a, b, c are in G. P., then $a + b$, $2b$, $b + c$ are in H. P.

4. If a, b, c are in H. P., then a, $a - c$, $a - b$ are in H. P.

5. If x^2, y^2, z^2 are in A. P., then $y + z$, $z + x$, $x + y$ are in H. P.

6. If a, b, c are in H. P., then $2a - b$, b, $2c - b$ are in G P.

7. If a, b, c are in A. P. and b, c, d are in H. P., then $a : b = c : d$

8. If three distinct numbers a, b, c are in A. P., while a^2, b^2, c^2 are in H P., then $-\frac{a}{2}$, b, c are in G. P.

9. Insert four harmonical means between $\frac{2}{3}$ and $\frac{2}{13}$; between 2 and $\frac{24}{7}$

CHAPTER VII.

COMPOUND INTEREST AND ANNUITIES.

71. Problem: To find the amount A of a given sum or principal P at compound interest for n years at the rate of r per cent a year.

Since the interest on P for one year is Pr, the amount of both principal and interest at the end of the first year will be $P + Pr = P(1 + r)$. At the end of the second year, the amount will be $\{P(1 + r)\}(1 + r) = P(1 + r)^2$. At the end of the third year, the amount will be $P(1 + r)^3$, etc. Hence at the end of n years, the amount will be

$$A = P(1 + r)^n.$$

If the interest be compounded semi-annually, the amount at the end of n years is found, by a similar argument, to be

$$A_2 = P\left(1 + \frac{r}{2}\right)^{2n}.$$

If the interest be compounded quarterly, the amount will be

$$A_4 = P\left(1 + \frac{r}{4}\right)^{4n}.$$

Example. Find the amount of $1000 in 20 years at compound interest at 6 per cent a year, payable semi-annually.

Since $P = 1000$, $n = 20$, $r = .06$, the amount is $A_2 = 1000(1.03)^{40}$.

By the four-place logarithmic table, pp. 24. 25, we have

$$\log 1.03 = 0.0128.$$

$$\therefore \log A_2 = \log 1000 + 40 \log 1.03 = 3.5120.$$

$$\therefore A_2 = 3251.$$

the last figure being wholly unreliable. With a seven-place table,

$$\log 1.03 = 0.0128372,$$

and $\log A_2 = 3.5134880$, whence $A_2 = 3262.03$.

72. Problem: To find the present value P of a sum A which is to be paid at the end of n years, allowing compound interest at r per cent.

The amount of P at the end of n years must equal A. Hence

$$P = A(1+r)^{-n}.$$

73. A fixed sum of money paid annually is called an **annuity.** The subject finds immediate application in Insurance. The simplest problem in annuities is the following:

To find the amount A of an annuity a allowed to accumulate for n years at compound interest at r per cent.

The first annual installment a is due at the end of the first year and hence draws interest for $n-1$ years; its amount will be $a(1+r)^{n-1}$. The second annual installment a will amount to $a(1+r)^{n-2}$, etc. The last installment a draws no interest. Hence the amount of the n installments will be

$$A = a + a(1+r) + a(1+r)^2 + \dots + a(1+r)^{n-2} + a(1+r)^{n-1}$$

$$= a\left\{\frac{(1+r)^n - 1}{(1+r) - 1}\right\} = \frac{a}{r}\left\{(1+r)^n - 1\right\},$$

by the formula for the sum of geometrical progression of ratio $1+r$.

74. Problem: To find the present value of an annuity a to continue n years, allowing compound interest at r per cent.

By § 72, the present value of a due one year hence is $a(1+r)^{-1}$; the present value of a due two years hence is $a(1+r)^{-2}$; . . .; the present value of a due n years hence is $a(1+r)^{-n}$. The total present value is

$$a(1+r)^{-1} + a(1+r)^{-2} + \dots + a(1+r)^{-n}$$

$$= a(1+r)^{-1}\left\{\frac{1-(1+r)^{-n}}{1-(1+r)^{-1}}\right\} = \frac{a}{r}\left\{1-(1+r)^{-n}\right\}.$$

Since the ratio $(1+r)^{-1}$ is less than unity, the geometrical progression may be summed to infinity, giving $\frac{a}{r}$. Hence the present value of a *perpetuity* (perpetual annuity) a is $\frac{a}{r}$, the rate of compound interest being r. Thus a perpetuity of $100 has the present value $\frac{100}{.04} = 2500$ dollars, if compound interest is reckoned at 4 per cent.

EXERCISES.

[The calculation of $(1+r)^n$ for r small and n large by means of our four-place table of logarithms is not very accurate. By a six-place table,

$\log 1.02 = .008600$, $\log 1.03 = .012837$, $\log 1.04 = .017033$,
$\log 1.05 = .021189$, $\log 1.06 = .025306$, $\log 1.025 = .010724$.]

1. Find the amount of $1000 in 30 years at 6 per cent compound interest, the interest being paid annually.

2. What sum of money will amount to $900 in 7½ years at 4 per cent compound interest, interest being compounded semi-annually?

3. What is the present worth of $5000 due in 10 years, allowing compound interest at 5 per cent paid semi-annually?

4. In how many years will a sum of money double itself at 5 per cent compound interest, due annually?

5. Find the amount of an annuity of $1000 in 20 years, allowing compound interest at 3 per cent per annum.

6. What is the present value of an annuity of $500 for 30 years, allowing interest at 5 per cent compounded semi-annually?

7. A person borrows $10000 at 5 per cent compound interest. How much must he pay in annual installments in order that the whole debt may be paid in 15 years?

8. What should I pay for a perpetuity of $500 to begin 10 years hence, if compound interest is reckoned at 3 per cent?

CHAPTER VIII.

UNDETERMINED COEFFICIENTS ; PARTIAL FRACTIONS.

75. A **variable** quantity is one which may assume different values (usually an unlimited number of values) in the same investigation. A **constant** quantity is one which retains the same value throughout the discussion.

Thus in a given geometrical progression whose first term is a and common ratio is r, the nth term t_n is ar^{n-1}. We consider a and r to be constants, while n is a variable ; by giving to n the values 1, 2, 3 4, . . . , in succession, we obtain for t_n the values a, ar, ar^2, ar^3, . . . , respectively.

The area of a circle of radius R is πR^2. To obtain different circles we let R vary; but π is a constant number.

A variable y is called a **function** of a variable x if to every value that may be assigned to x there corresponds a definite value of y. It sometimes happens that certain isolated values may not be assigned to x, since y then ceases to have a definite value (see § 100).

In the preceding examples, $t_n \equiv ar^{n-1}$ is a function of n, the area πR^2 of the circle is a function of the radius R. Similarly x^2, $3x + 7$, $\sqrt[3]{x}$, $\log x$ are functions of x.

It is often convenient to employ symbols which emphasize the functional dependence. Thus, for a and r constant and n variable, we used t^n to denote the function ar^{n-1} of n. Likewise, $\sqrt{x}$ and $\log x$ are functional symbols.

76. By formulæ (8) and (9) of § 31, we have the identity

$$(1)\qquad ax^2+bx+c\equiv a(x-\alpha)(x-\beta),$$

if α and β denote the roots of the equation $ax^2+bx+c=0$.

To generalize this result, consider the expression

$$E_x\equiv ax^n+bx^{n-1}+cx^{n-2}+\ldots+kx+l,$$

called a **rational integral function** of x of degree n. If the expression E_x vanishes when α is substituted for x, then, by the factor theorem, $x-\alpha$ is a factor of E_x. The first term of the quotient Q obtained upon dividing E_x by $x-\alpha$ is clearly ax^{n-1}. Hence

$$E_x=(x-\alpha)Q,\quad Q\equiv ax^{n-1}+\ldots$$

Similarly, if Q vanishes when β is written for x, then

$$Q=(x-\beta)Q',\quad Q'\equiv ax^{n-2}+\ldots$$

Hence $$E_x=(x-\alpha)(x-\beta)Q',$$

so that E_x vanishes when β is substituted for x. *If** there be n distinct values $\alpha, \beta, \ldots, \nu$ of x for which E_x vanishes, a continuation of the preceding argument shows that E_x has the n factors $x-\alpha$, $x-\beta$, . . . , $x-\nu$ and a final numerical factor a, so that

$$(2)\qquad E_x=a(x-\alpha)(x-\beta)\ldots(x-\nu).$$

Then $\alpha, \beta, \ldots, \nu$ are roots of the equation $E_x=0$.

77. Theorem. *A rational integral function of the nth degree in x cannot vanish for* MORE THAN *n values of x, unless the coefficients of all the powers of x are zero.*

Suppose that the function E_x vanishes for $n+1$ distinct values λ and $\alpha, \beta \ldots, \nu$ of x. Then E_x must have the form (2), and must moreover vanish for $x=\lambda$. Hence

$$a(\lambda-\alpha)(\lambda-\beta)\ldots(\lambda-\nu)=0.$$

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* The question of the existence of such values $\alpha, \beta, \ldots$, is considered in Chapter XIX.

Since the differences $\lambda - \alpha, \ldots, \lambda - \nu$ all differ from zero, a must be zero. Hence E_x reduces to $bx^{n-1} + cx^{n-2} + \ldots$ Since this function of degree $n - 1$ vanishes for more than $n - 1$ values, b must be zero. Similarly, $c = 0, \ldots, k = 0, l = 0$.

Another statement of the theorem is the following:

An equation of degree n cannot have more than n roots unless all the coefficients are zero.

78. Suppose that the two functions of degree n

$$p_0x^n + p_1x^{n-1} + \ldots + p_{n-1}x + p_n, \quad q_0x^n + q_1x^{n-1} + \ldots + q_{n-1}x + q_n$$

are equal in value for more than n values of x. Then

$$(p_0 - q_0)x^n + (p_1 - q_1)x^{n-1} + \ldots + (p_{n-1} - q_{n-1})x + (p_n - q_n)$$

vanishes for more than n values of x. By the previous theorem, the coefficients of all the powers of x must be zero, whence

$$p_0 = q_0, \quad p_1 = q_1, \quad \ldots, \quad p_{n-1} = q_{n-1}, \quad p_n = q_n.$$

If two rational integral functions of the nth degree in x be equal in value for more than n values of x, the coefficients of like powers of x are equal.

In particular, if two rational integral functions of x each of finite degree are equal in value for *all values* of x, the coefficients of like powers of x may be equated to each other.

79. To illustrate the application of the preceding theorems to problems on integral functions, we consider certain examples.

EXAMPLE 1. Find the conditions on a, b, c, d in order that $ax^3 + bx^2 + cx + d$ may be a perfect cube for all values of x.

It must be the cube of an expression $lx + m$ of the first degree, in which l and m are, as yet, **undetermined coefficients.** Set *

$$ax^3 + bx^2 + cx + d \equiv (lx + m)^3 \equiv l^3x^3 + 3l^2mx^2 + 3lm^2x + m^3.$$

Since the first and third expressions are equal for all values of x,

$$a = l^3, \quad b = 3l^2m, \quad c = 3lm^2, \quad d = m^3.$$

Hence $l = a^{\frac{1}{3}}$, $m = d^{\frac{1}{3}}$. The second and third conditions then give

$$b = 3a^{\frac{2}{3}}d^{\frac{1}{3}}, \quad c = 3a^{\frac{1}{3}}d^{\frac{2}{3}}.$$

---

* The symbol $\equiv$ is employed in identities. Here the relation is an *identity in x*.

Inversely, if b and c have these values, then

$$ax^3 + bx^2 + cx + d = (a^{\frac{1}{3}}x + d^{\frac{1}{3}})^3.$$

EXAMPLE 2. Determine the value of k so that $x^4 + 4x^2 + 3x + k$ shall be divisible by $x^2 + x + 1$.

The quotient must have the form $x^2 + hx + k$. Set

$$x^4 + 4x^2 + 3x + k \equiv (x^2 + x + 1)(x^2 + hx + k).$$

Expanding the right member and equating coefficients of like powers of x in the two expressions, we have the conditions

$$0 = h + 1, \quad 4 = k + h + 1, \quad 3 = h + k,$$

which are all satisfied if $h = -1$, $k = 4$.

EXAMPLE 3. Find the sum of the squares of the first n integers.

Since the sum depends upon n, we assume that

$$1^2 + 2^2 + \ldots + (n-1)^2 + n^2 \equiv a + bn + cn^2 + dn^3 + en^4 + fn^5 + \ldots,$$

where $a, b, c, d, e, f, \ldots$ are undetermined coefficients, each independent of n. Changing n into $n + 1$, we get

$$1^2+2^2+\ldots+n^2+(n+1)^2 \equiv a+b(n+1)+c(n+1)^2+d(n+1)^3+e(n+1)^4+\ldots$$

Subtracting the previous identity, we get

$$n^2 + 2n + 1 \equiv b + 2cn + c + 3dn^2 + 3dn + d + 4en^3 + 6en^2 + 4en + e + \ldots$$

Since this equation holds for every value of n, the coefficients of like powers of n may be equated. Hence $e = 0$ and f, and all succeeding coefficients, must be zero. Also,

$$3d = 1, \quad 3d + 2c = 2, \quad d + c + b = 1,$$

whence $d = \frac{1}{3}$, $c = \frac{1}{2}$, $b = \frac{1}{6}$. Hence

$$1^2 + 2^2 + \ldots + (n-1)^2 + n^2 \equiv a + \tfrac{1}{6}n + \tfrac{1}{2}n^2 + \tfrac{1}{3}n^3.$$

If $n = 1$, there is a single term 1 in the series, so that

$$1 = a + \tfrac{1}{6} + \tfrac{1}{2} + \tfrac{1}{3}, \quad \text{or} \quad a = 0.$$

Hence $$1^2 + 2^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

EXERCISES.

1. When is $a^2x^4 + bx^3 + cx^2 + dx + f^2$ a perfect square?

2. Find the condition that $x^3 - 3px + 2q$ may have a factor of the form $(x - c)^2$.

3. Find k so that there are solutions x, y, z not all zero of

$$x + y - z = 0, \quad 2x - y + 3z = 0, \quad x + ky + z = 0.$$

4. If $ax^3 + bx^2 + cx + d$ is divisible by $x^2 + h^2$, then $ad = bc$.

Using undetermined coefficients, establish the sums

5. $1 + 2 + 3 + \ldots + n = \frac{1}{2}n(n+1)$.

6. $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{1}{3}n(4n^2 - 1)$.

7. $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{1}{4}n^2(n + 1)^2$.

8. $1^3 + 3^3 + 5^3 + \ldots + (2n - 1)^3 = n^2(2n^2 - 1)$.

9. $1^4 + 2^4 + 3^4 + \ldots + n^4 = \frac{1}{30}n(n + 1)(2n + 1)(3n^2 + 3n - 1)$.

10. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n + 1) = \frac{1}{3}n(n + 1)(n + 2)$.

11. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \ldots + n(n + 1)(n + 2) = \frac{1}{4}n(n+1)(n+2)(n+3)$.

12. The sum of the cubes of the first n integers equals the square of their sum.

PARTIAL FRACTIONS.

80. It is desirable to be able to express a complex fraction as the sum of simpler fractions, called **partial fractions.** Thus

$$\frac{4 - 5x}{1 - 3x + 2x^2} = \frac{1}{1 - x} + \frac{3}{1 - 2x}. \tag{3}$$

To expand the complex fraction in the left member into a series of ascending powers of x, we may expand the partial fractions on the right and add the resulting series. Indeed, by the theory of geometrical progressions, if r be numerically less than unity, the following expansion holds:

$$\frac{1}{1 - r} = 1 + r + r^2 + r^3 + \ldots$$

Hence, if $2x$ is numerically less than unity,

$$\frac{1}{1 - x} = 1 + x + x^2 + \ldots + x^n + \ldots,$$

$$\frac{3}{1 - 2x} = 3(1 + 2x + 4x^2 + \ldots + 2^n x^n + \ldots),$$

$$\frac{4 - 5x}{1 - 3x + 2x^2} = 4 + 7x + 13x^2 + \ldots + (3.2^n + 1)x^n + \ldots$$

81. To explain the general method employed to decompose into partial fractions a given complex fraction whose denominator is of higher degree in x than the numerator, consider the fraction in the left member of (3). We observe that the fractions $\frac{a}{1 - x}$ and

$\dfrac{b}{1-2x}$, when reduced to the denominator $(1-x)(1-2x)$, will contribute to the formation of a complex fraction with the given denominator and with a numerator of the first degree, while no new fraction with a denominator of the first degree will have this property. Hence if our aim is to be attained, the decomposition must be of the form

$$\frac{4-5x}{1-3x+2x^2}=\frac{a}{1-x}+\frac{b}{1-2x},$$

in which the undetermined coefficients a and b are to be determined. If this be possible, we shall have

$$4-5x=a(1-2x)+b(1-x)\equiv a+b-x(2a+b).$$

In view of its origin, this relation is to be true for all values of x except 1 and $\frac{1}{2}$. Then, by § 78, the relation must be true for all values of x, and the constant terms as well as the coefficients of x may be equated. Hence

$$a+b=4, \quad 2a+b=5.$$

Hence $a=1$, $b=3$, giving the true result (3).

82. As a second example, we decompose into partial fractions

$$\frac{4x^2+3x-1}{(x-1)^2(x+2)}.$$

It is clear that fractions with the denominators $x+2$, $x-1$, $(x-1)^2$ and with constants for numerators may contribute to the formation of a fraction with the denominator $(x+2)(x-1)^2$ and with a numerator of degree at most two. There is nothing gained by employing also fractions of the forms

$$\frac{mx+n}{(x+2)(x-1)}, \quad \frac{rx+s}{(x-1)^2},$$

since they are themselves expressible as sums of partial fractions,

$$\frac{\frac{1}{3}(m+n)}{x-1}+\frac{\frac{1}{3}(2m-n)}{x+2}, \quad \frac{r}{x-1}+\frac{s+r}{(x-1)^2},$$

respectively. Moreover, $\frac{1}{(x-1)^2}$ cannot be expressed as the sum of fractions with denominators of the first degree. The decomposition will therefore give partial fractions of simpler form than the given fraction only when

$$\frac{4x^2+3x-1}{(x-1)^2(x+2)}=\frac{a}{x+2}+\frac{b}{x-1}+\frac{c}{(x-1)^2},$$

for suitable values of the undetermined coefficients a, b, c. Then

$$(4)\quad 4x^2+3x-1=a(x-1)^2+b(x-1)(x+2)+c(x+2).$$

Equating the coefficients of x^2, of x, and the constant terms, we get

$$a+b=4,\quad -2a+b+c=3,\quad a-2b+2c=-1.$$

Solving by determinants or otherwise, we get $a=1$, $b=3$, $c=2$.

A simpler method of determining a, b, c is to substitute special values for x in (4). We observe that, in view of § 78, relation (4) is an identity and is therefore true for the values $x=1$, -2, etc. For $x=1$, we get $6=3c$; for $x=-2$, $9=9a$; for $x=0$, $-1=a-2b+2c$. Hence $c=2$, $a=1$, $b=3$.

83. The presence of imaginary numbers offers no difficulty. To decompose the following fraction, we set

$$\frac{42-19x}{(x-4)(x^2+1)}=\frac{a}{x-4}+\frac{b}{x+\sqrt{-1}}+\frac{c}{x-\sqrt{-1}}.$$

$$\therefore\quad 42-19x=a(x^2+1)+b(x-4)(x-\sqrt{-1})+c(x-4)(x+\sqrt{-1}).$$

Equating coefficients, we obtain the values of a, b, c given below. With the explanation made at the end of § 82, we may simplify the work. For $x=4$, we get $-34=17a$, whence $a=-2$. For $x=\sqrt{-1}$, we get

$$42-19\sqrt{-1}=c(\sqrt{-1}-4)(2\sqrt{-1}),\quad c=1+\tfrac{11}{2}\sqrt{-1}.$$

For $x = -\sqrt{-1}$, we get $b = 1 - \frac{11}{2}\sqrt{-1}$. Hence

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{-2}{x-4} + \frac{1-\frac{11}{2}\sqrt{-1}}{x+\sqrt{-1}} + \frac{1+\frac{11}{2}\sqrt{-1}}{x-\sqrt{-1}}.$$

The second and third partial fractions are conjugates, since one is derived from the other by replacing $\sqrt{-1}$ by $-\sqrt{-1}$. Hence their sum must be real (§ 4). In fact, their sum is $\dfrac{2x-11}{x^2+1}$.

We might have avoided the introduction of imaginaries by setting

$$\frac{42-19x}{(x-4)(x^2+1)} = \frac{a}{x-4} + \frac{ex+f}{x^2+1}.$$

$$\therefore \quad 42 - 19x = a(x^2+1) + (ex+f)(x-4).$$

Observing that this relation is an identity by § 78, we may set $x = 4$, whence $a = -2$. Equating the coefficients of x^2 and of x, we get

$$e = 2, \quad -19 = -4e + f, \quad \text{whence } f = -11.$$

84. If the degree of the numerator of the given fraction equals or exceeds the degree of the denominator, we first divide the numerator by the denominator and obtain a remainder of degree less than the degree of the denominator. For example,

$$\frac{12x^3+10x^2-14}{3x^2-2x-1} = 4x + 6 + \frac{16x-8}{3x^2-2x-1}.$$

Proceeding as usual, the second fraction may be decomposed into

$$\frac{10x}{3x+1} + \frac{2}{x-1}.$$

The division mentioned may be avoided by assuming that

$$\frac{12x^3+10x^2-14}{3x^2-2x-1} = 4x + a + \frac{b}{3x+1} + \frac{c}{x-1},$$

and determining a, b, c by the usual method.

EXERCISES.

Resolve into partial fractions:

1. $\dfrac{7x-1}{x^2-1}$. **2.** $\dfrac{x^2+1}{x(x-1)^2}$. **3.** $\dfrac{5-13x}{1-5x+6x^2}$.

4. $\dfrac{x^2+15x+8}{(x-5)(x+1)^2}$. **5.** $\dfrac{x^2+8x}{(x+5)(x^2+1)}$. **6.** $\dfrac{5x^2-x+2}{x^3+x}$.

7. $\dfrac{x^2-3x+6}{x^2-3x+2}$. **8.** $\dfrac{1+5x}{1-x^3}$. **9.** $\dfrac{2x^3-2}{x^3+1}$.

10 $\dfrac{x^4+x^2+2}{x^4+1}$. **11.** $\dfrac{2x^2-x+1}{(x-1)^3}$. **12.** $\dfrac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$.

13. Expand into series the fractions of Exs. 1, 3, 5, 7, 9.

CHAPTER IX.

PERMUTATIONS AND COMBINATIONS; BINOMIAL AND MULTINOMIAL THEOREMS FOR POSITIVE INTEGRAL INDEX.

85. Three letters a, b, c may be arranged in six ways

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

While there are only three groups (or selections) each of two letters chosen from three letters a, b, c, viz., ab, ac, bc, we obtain six arrangements of three letters two at a time, namely,

$$ab, \quad ba, \quad ac, \quad ca, \quad bc, \quad cb.$$

Definitions. Each of the arrangements which can be made with r things chosen from n things is called **a permutation of the n things r at a time.** Each of the groups (selections) which can be made by selecting r things from n things is called a **combination of the n things r at a time.**

Thus there are six permutations of a, b, c two at a time:

$$ab, \quad ba, \quad ac, \quad ca, \quad bc, \quad cb.$$

There are three combinations of a, b, c two at a time: ab, ac, bc.

In a combination, the order in which the letters are written is indifferent; in a permutation, the order is essential. Thus the pairs ab and ba give the same combination, but give distinct permutations.

86. *The number of permutations of n different things r at a time is* $n(n-1)(n-2)\ldots(n-r+1)$.*

We are to find the number of ways in which we can fill r places when we have n different things at our disposal.

For the first place we may take any one of the n things; for the second place any one of the remaining $n-1$ things; for the third place any one of the now remaining $n-2$ things, etc. Hence the r places may be filled in $n(n-1)(n-2)\ldots(n-r+1)$ ways, since the rth factor is $n-(r-1)$.

Corollary. The number of permutations of n different things taken all at a time is $n(n-1)(n-2)\ldots 3\cdot 2\cdot 1$. This product of all the natural numbers up to and including n will be denoted by the symbol $n!$ which is read "n factorial." †

For example, there are $3! = 6$ permutations of a, b, c three at a time (given in § 85) and $3\cdot 2 = 6$ permutations of a, b, c two at a time (given in § 85).

87. We notice that each of the three combinations *ab, ac, bc* of the letters a, b, c taken two at a time furnishes exactly two permutations of a, b, c two at a time; thus the combination *ab* furnishes the two permutations *ab* and *ba*. Moreover, the six resulting permutations give *all* the permutations of a, b, c two at a time. Similar remarks hold true in the general case next considered.

Let ${}_nC_r$ denote the number of combinations of n different things r at a time. Let ${}_nP_r$ denote the number of permutations of n different things r at a time. We have shown that

$$ {}_nP_r = n(n-1)(n-2)\ldots(n-r+1). $$

Each one of the ${}_nC_r$ combinations consists of a set of r different things, which may therefore be arranged or permuted in exactly ${}_rP_r \equiv r!$ distinct ways. Every such arrangement is a permutation of the n things taken r at a time. By starting with a suitable combination of r of the things and arranging them in a suitable way, we may reach any given permutation of the n things r at a time. Hence

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* The number is zero if $r > n$. The product then has a factor $n - n = 0$.

† The symbol $\underline{|n}$ is also used.

$$ {}_nC_r \times r! = {}_nP_r = n(n-1)\ldots(n-r+1). $$

$$ \therefore\ {}_nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot 2\cdot 3\ldots r}. $$

Upon multiplying the numerator and the denominator by $(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1 \equiv (n-r)!$, we get*

$$ {}_nC_r = \frac{n!}{r!\,(n-r)!}. $$

88. *The number of combinations of n different things r at a time equals the number of combinations of n things $n-r$ at a time.*

When a set of r things is selected from n things, there is left a set of $n-r$ things. Moreover, any given set of $n-r$ things may be left (negative selection) by making a suitable selection of r things. Hence ${}_nC_r = {}_nC_{n-r}$.

The proposition also follows from the formula for ${}_nC_k$. Thus

$$ {}_nC_{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!} = \frac{n!}{(n-r)!\,r!} = {}_nC_r. $$

89. Theorem. *For positive integers n and r,*

$$ {}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r. $$

The combinations of $n+1$ different letters $a_1, a_2, \ldots, a_n, a_{n+1}$ r at a time may be separated into two sets, according as the combination contains the last latter a_{n+1} or does not. If the last letter be taken, there remain only $r-1$ letters to be selected from the n letters $a_1, a_2, \ldots, a_n$, which can be done in ${}_nC_{r-1}$ ways. If the last letter be not taken, we must select r letters from the n letters $a_1, a_2, \ldots, a_n$, which can be done in ${}_nC_r$ ways. The sum of the two numbers ${}_nC_{r-1}$ and ${}_nC_r$ must equal ${}_{n+1}C_r$.

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* For $r = n$, the first result gives ${}_nC_n = \frac{n!}{n!} = 1$, while the second gives ${}_nC_n = \frac{n!}{n!\,0!}$. The second formula therefore holds for $r = n$ only when we adopt the notation $0! \equiv 1$. Similarly, the result in § 88 holds for $r = n$ only when we put ${}_nC_0 \equiv 1$.

The formula remains true for $r = 1$, if we take ${}_nC_0 = 1$, there being one way of selecting no objects from n objects.

An interesting application is the construction of Pascal's Triangle:

$$\begin{array}{lllll}
{}_0C_0 & & & & \\
{}_1C_0 & {}_1C_1 & & & \\
{}_2C_0 & {}_2C_1 & {}_2C_2 & & \\
{}_3C_0 & {}_3C_1 & {}_3C_2 & {}_3C_3 & \\
{}_4C_0 & {}_4C_1 & {}_4C_2 & {}_4C_3 & {}_4C_4
\end{array}
\qquad\qquad
\begin{array}{lllll}
1 & & & & \\
1 & 1 & & & \\
1 & 2 & 1 & & \\
1 & 3 & 3 & 1 & \\
1 & 4 & 6 & 4 & 1
\end{array}$$

The above formula now shows that any number ${}_{n+1}C_r$ equals the sum of the number ${}_nC_r$ just above it and the number ${}_nC_{r-1}$ to the left of ${}_nC_r$. For example, ${}_4C_2 = {}_3C_2 + {}_3C_1$, or $6 = 3 + 3$. The numbers in the next row of the table on the right would therefore be

$$1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1.$$

It appears that the numbers in the nth row of the table are the binomial coefficients in the expansion of $(1 + x)^n$ [§ 91].

90. We proceed to determine the number of permutations of n things taken all at a time, when the things are not all different. There are only three permutations of a, a, b taken three at a time, namely, aab, aba, baa. If we replace the two a's by two distinct letters a_1 and a_2, the first permutation aab will furnish two permutations a_1a_2b and a_2a_1b. In this way, we reach the $3 \cdot 2 = 6$ permutations of a_1, a_2, b.

In general, let there be n letters, p of which are a's, q of which are b's, and r of which are c's, so that $n = p + q + r$. Pet P denote the required number of permutations of these n letters n at a time. Consider any one of the permutations as

$$aa \ldots abb \ldots bcc \ldots c.$$

If we replace the p letters a by p distinct letters $a_1, a_2, \ldots, a_p$, we may derive from the given permutation $p!$ permutations

$$a_1a_2 \ldots a_pb \ldots bc \ldots c, \quad a_2a_1 \ldots a_pb \ldots bc \ldots c, \text{ etc.}$$

Hence from the P permutations of the a's, b's, c's, we derive $P \,.\, p!$ permutations of $a_1, a_2, \ldots, a_p, b, \ldots, b, c, \ldots, c$. Similarly, if in one of the new permutations we replace the q letters b by q distinct letters $b_1, b_2, \ldots, b_q$, we may obtain exactly $q!$ permutations

$$a_1a_2 \ldots a_pb_1b_2 \ldots b_qc \ldots c, \quad a_1a_2 \ldots a_pb_2b_1 \ldots b_qc \ldots c, \text{ etc.}$$

Hence the $P \,.\, p!$ permutations of $a_1, a_2, \ldots, a_p, b, \ldots, b, c, \ldots, c$ give rise to exactly $P \,.\, p!\,q!$ permutations of $a_1, a_2, \ldots, a_p, b_1, \ldots, b_q, c, \ldots, c$. Finally, if the r letters c be replaced by r distinct letters $c_1, c_2, \ldots, c_r$, the resulting number of permutations on the $p + q + r \equiv n$ distinct letters will be $P \,.\, p!\,q!\,r!$ But the number of permutations of n distinct letters n at a time is $n!$ Hence

$$P \,.\, p!\,q!\,r! = n! \qquad P = \frac{n!}{p!\,p!\,r!}.$$

By a similar proof, the number of permutations of n things n at a time, p of which are alike, q alike, r alike, . . . , t alike, is seen to be

$$\frac{n!}{p!\,q!\,r! \ldots t!} \qquad (n \equiv p + q + r + \ldots + t).$$

For example, the number of permutations of all the letters of the word *Mississippi* is $\dfrac{11!}{4!\,4!\,2!\,1!} = 34650$.

EXERCISES.

1. In how many different ways can five boys stand in a row ?

2. How many numbers of six digits can be formed by using the numbers 1, 2, 3, 4, 5, 6 ? How many numbers of four digits ?

3. How many numbers of three digits (the first not zero) can be formed with 0, 1, 2, 3, 4, 5 ? How many of five digits ?

4. Find the number of permutations of all the letters of the word *animal*; of the word *relative*. In how many permutations do the vowels and consonants alternate? How many of the latter end with a vowel?

5. How many numbers less than 1000 can be made with the digits 0, 1, 2, 3, 4, 5, 6? How many with the digits 1, 2, 3, 4, 5, 6, 7?

6. In how many ways can seven boys form a ring? a row?

7. How many sums of money can be made with 5 pennies, 3 dimes, 2 quarters, and 1 dollar?

8. Prove that ${}_{n+2}C_{r+1} = {}_nC_{r+1} + 2\,{}_nC_r + {}_nC_{r-1}$.

9. If ${}_nC_5 = {}_nC_{10}$, find ${}_nC_3$.

10. In how many ways can 4 men and 2 boys be chosen from 12 men and 5 boys?

11. In how many ways can six gentlemen and six ladies be arranged at a round table so that no two gentlemen are adjacent?

12. How many triangles can be formed by joining three of the vertices of a general octagon?

13. Prove that ${}_nC_{r+1} = {}_{n-1}C_r + {}_{n-2}C_r + {}_{n-3}C_r + \dots + {}_rC_r$.

14. By considering the combinations of s letters some of which are chosen from $a_1, a_2, \dots, a_p$, and the others from $b_1, b_2, \dots, b_q$, show that

$${}_{p+q}C_s = {}_pC_s + {}_pC_{s-1} \cdot {}_qC_1 + {}_pC_{s-2} \cdot {}_qC_2 + \dots + {}_pC_1 \cdot {}_qC_{s-1} + {}_qC_s.$$

BINOMIAL THEOREM FOR POSITIVE INTEGRAL INDEX.

91. Let n be a positive integer and consider the product

$$(a+b)(a+b)\dots(a+b) \qquad [n \text{ factors}].$$

One term of the product is a^n; it is obtained by taking the letter a from each parenthesis. There will be n terms $a^{n-1}b$, since the letter b may be chosen from any one of the n parentheses, which may be done in ${}_nC_1 \equiv n$ ways. There will be ${}_nC_2$ terms $a^{n-2}b^2$, since two b's may be chosen from any two of the n parentheses and the a's from the remaining parentheses. In general, there will be ${}_nC_r$ terms $a^{n-r}b^r$, since the r b's may be chosen from any r of the n parentheses, the a's being then taken from the remaining parentheses. Hence

$$(a+b)^n = a^n + {}_nC_1a^{n-1}b + {}_nC_2a^{n-2}b^2 + \dots + {}_nC_ra^{n-r}b^r + \dots + b^n.$$

The expression on the right is called the **expansion** of $(a+b)^n$, and the theorem involved is called the **binomial theorem** for the case in which the index n is a positive integer. The $r+1$st term is ${}_nC_ra^{n-r}b^r$; it is called the **general term of the expansion.**

Substituting for ${}_nC_1$, ${}_nC_2$, . . . their values (§ 87), we get

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \dots$$

$$+\frac{n(n-1)\dots(n-r+1)}{1\cdot 2\dots r}a^{n-r}b^r + \dots + nab^{n-1} + b^n.$$

By § 88, ${}_nC_r = {}_nC_{n-r}$ for all values of r from 1 to n. Hence *in the expansion of $(a+b)^n$, the coefficients of any two terms equidistant from the beginning and the end are equal.*

92. To determine the greatest coefficient, we observe that

$${}_nC_r = \frac{n(n-1)\dots(n-r+2)(n-r+1)}{1\cdot 2\dots r-1\cdot r},$$

$${}_nC_{r-1} = \frac{n(n-1)\dots(n-r+2)}{1\cdot 2\dots r-1}.$$

$$\therefore \quad {}_nC_r = {}_nC_{r-1}\cdot\frac{n-r+1}{r}.$$

Hence ${}_nC_r \gtreqless {}_nC_{r-1}$ according as $\frac{n-r+1}{r} \gtreqless 1$, namely, according as $r \lesseqgtr \frac{1}{2}(n+1)$, since the inequality sign must be reversed when the signs of the two members are changed. Hence the number ${}_nC_r$ increases as r increases, so long as r is less than $\frac{1}{2}(n+1)$, but decreases as r increases when $r > \frac{1}{2}(n+1)$. The case $r = \frac{1}{2}(n+1)$ occurs only when n is odd; then $r-1 = n-r$, so that ${}_nC_r = {}_nC_{r-1}$, in agreement with the result of § 88. Hence, if n is odd, the coefficients ${}_nC_{\frac{1}{2}(n+1)}$ and ${}_nC_{\frac{1}{2}(n-1)}$ are equal and are greater than the remaining coefficients; if n is even, the coefficient ${}_nC_{\frac{1}{2}n}$ is the greatest coefficient.

Observe that the number of terms in the expansion of $(a+b)^n$ is $n+1$, and that, for n even, ${}_nC_{\frac{1}{2}n}$ is the coefficient of the $\frac{1}{2}n+1$st term, which is then *the* middle term; while for n odd, there is no middle term, but there are two terms, the $\frac{1}{2}(n+1)$st and the $\frac{1}{2}(n+3)$rd, at the middle of the expansion, their coefficients being ${}_nC_{\frac{1}{2}(n-1)}$ and ${}_nC_{\frac{1}{2}(n+1)}$. We may state the result:

In the expansion of $(a+b)^n$, the greatest coefficient is the middle term if n be even; while for n odd the middle pair of terms have the greatest coefficients.

EXERCISES.

1. Find the greatest coefficient in the expansion of $(a+b)^6$.

2. Find the greatest coefficients in the expansion of $(a+b)^7$.

3. Two coefficients ${}_nC_r$ and ${}_nC_s$ are equal only when $r+s=n$

4. If the 10th and 12th coefficients are equal, find the 4th.

5. If ${}_{2n}C_3 = 24\,{}_nC_4$, find n.

6. The sum of the coefficients in the expansion of $(a+b)^n$ is 2^n.

7. The sum of the coefficients of the odd terms of a binomial expansion equals the sum of the coefficients of the even terms.

8. If there be a middle term, its coefficient will be *even*.

9. ${}_nC_1 + 2\,{}_nC_2 + 3\,{}_nC_3 + \ldots + r\,{}_nC_r + \ldots + n\,{}_nC_n = n\,2^{n-1}$.

10. ${}_nC_1 - 2\,{}_nC_2 + 3\,{}_nC_3 - \ldots + (-1)^{n-1}n\,{}_nC_n = 0$.

11. ${}_nC_0 + \frac{1}{3}\,{}_nC_2 + \frac{1}{5}\,{}_nC_4 + \frac{1}{7}\,{}_nC_6 + \ldots = \dfrac{2^n}{n+1}$.

12. ${}_nC_0\,{}_nC_1 + {}^nC_1\,{}_nC_2 + {}_nC_2\,{}_nC_3 + \ldots + {}_nC_{n-1}\,{}_nC_n = \dfrac{(2n)\,!}{(n+1)\,!\,(n-1)\,!}$.

13. ${}_nC_1 - \frac{1}{2}\,{}_nC_2 + \frac{1}{3}\,{}_nC_3 - \ldots + (-1)^{n-1}\dfrac{1}{n}\,{}_nC_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

14. ${}_nC_0^2 + {}_nC_1^2 + {}_nC_2^2 + \ldots + {}_nC_n^2 = (2n)\,! \div (n\,!)^2$.

MULTINOMIAL THEOREM.

93. Let n be a positive integer and consider the product

$$(a+b+c+d)(a+b+c+d)\;\ldots\;(a+b+c+d) \qquad [n \text{ factors}].$$

Every term of the product is of the form

$$a^\alpha b^\beta c^\gamma d^\delta \qquad (\alpha+\beta+\gamma+\delta=n),$$

each exponent α, β, γ, δ being a positive integer or zero. But various terms will be equal and may be added. For α, β, γ, δ

given, the terms equal to $a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}$ may all be obtained by selecting the letter a from α parentheses, b from β parentheses, c from γ parentheses, and d from the remaining δ parentheses, the selection to be made in every possible way. By § 90, this selection may be done in exactly

$$\frac{n\,!}{\alpha\,!\,\beta\,!\,\gamma\,!\,\delta\,!}$$

ways. Hence the general term in the expansion of $(a+b+c+d)^n$ is

$$\frac{n\,!}{\alpha\,!\,\beta\,!\,\gamma\,!\,\delta\,!}a^{\alpha}b^{\beta}c^{\gamma}d^{\delta}.$$

EXAMPLE 1. Find the coefficient of $a^3b^2c^2$ in $(a+b+c)^7$

The coefficient is $\frac{7\,!}{3\,!\,2\,!\,2\,!} = 210.$

EXAMPLE 2. Find the coefficient of x^5 in $(a+bx+cx^2)^8$.

The general term of the expansion is

$$\frac{8\,!}{\alpha\,!\,\beta\,!\,\gamma\,!}a^{\alpha}b^{\beta}c^{\gamma}x^{\beta+2\gamma} \qquad (\alpha+\beta+\gamma=8).$$

We seek the terms in which $\beta+2\gamma=5$. Each of the quantities α, β, γ is to be a positive integer or zero. Hence $\gamma \leqq 2$. For $\gamma=2$, we get $\beta=1$, $\alpha=5$; for $\gamma=1$, $\beta=3$, $\alpha=4$; for $\gamma=0$, $\beta=5$, $\alpha=3$. The coefficient of x^5 is therefore

$$\frac{8\,!}{5\,!\,2\,!}a^5bc^2+\frac{8\,!}{4\,!\,3\,!}a^4b^3c+\frac{8\,!}{3\,!\,5\,!}a^3b^5$$

EXERCISES.

1. Find the coefficients of a^2b^2c and a^4b in $(a+b+c)^5$.
2. Find the coefficient of $a^2b^3c^4d$ in $(a+b+c+d)^{10}$.
3. Find the coefficient of x^4 in $(1+3x+4x^2)^9$.
4. Find the coefficient of x^5 in $(1+3x-x^2)^6$.
5. Find the coefficient of x^4 in $(1+2x+3x^2+4x^3)^5$.

CHAPTER X.

PROBABILITY (CHANCE).

94. Definitions. If an event can happen in m ways and fail in n ways, and if each of these $m + n$ ways is equally likely, the **probability**, or the **chance**, of its happening is $\frac{m}{m+n}$, and the **probability** of its failing is $\frac{n}{m+n}$.

An equivalent statement is that *the odds are m to n in favor of the event*, or *n to m against the event.*

For example, if a single ball is drawn at random from a bag containing 4 white and 6 black balls, the chance of drawing a white ball is $\frac{4}{10}$ and the chance of drawing a black ball is $\frac{6}{10}$. The chance that the ball will be either white or black is $\frac{4}{10} + \frac{6}{10} = 1$, namely, one of the events is certain to happen. *The probability 1 denotes certainty.*

95. EXAMPLE 1. From a bag containing 4 white and 6 black balls, 2 balls are drawn. Find the chance (1) that both are white; (2) that both are black; (3) that one is white and one black.

The total number of ways in which 2 balls can be drawn from 10 is ${}_{10}C_2 = 45$. The number of ways of drawing 2 white balls is ${}_4C_2 = 6$; of drawing 2 black balls ${}_6C_2 = 15$; of drawing 1 white and 1 black ball ${}_4C_1 \cdot {}_6C_1 = 24$. As a check, we note that $6 + 15 + 24 = 45$. The chances of the respective events are therefore $\frac{6}{45}$, $\frac{15}{45}$, $\frac{24}{45}$.

EXAMPLE 2. A man has 3 chances in a lottery in which there are 5 prizes and 10 blanks. Find his chance of drawing a prize.

Of the 15 tickets, 3 may be drawn in ${}_{15}C_3 = 455$ ways. 3 prize tickets may be drawn in ${}_5C_3 = 10$ ways; 2 prize tickets and 1 blank may be drawn in $_5C_2 \cdot {}_{10}C_1 = 100$ ways ; 1 prize ticket and 2 blanks in $5\,{}_{10}C_2 = 225$ ways ; 3 blanks in $_{10}C_3 = 120$ ways. The respective chances are

$$\tfrac{10}{455},\quad \tfrac{100}{455},\quad \tfrac{225}{455},\quad \tfrac{120}{455},$$

whose sum is 1, a check upon the work. The chance of drawing one or more prizes is the sum of the first three chances, viz., $\frac{335}{455}$.

EXAMPLE 3. If 5 coins are tossed, what is the probability that 3 heads and 2 tails will appear ?

As each coin may fall in 2 ways, the 5 coins may fall in $2^5 = 32$ ways. These events may be classified as follows : 5 heads may fall in 1 way ; 4 heads and 1 tail in 5 ways ; 3 heads and 2 tails in 10 ways ; 2 heads and 3 tails in 10 ways ; 1 head and 4 tails in 5 ways ; 5 tails in 1 way. As a check, we observe that $1 + 5 + 10 + 10 + 5 + 1 = 32$. The chance for 3 heads and 2 tails is therefore $\frac{10}{32}$.

EXAMPLE 4. If a pair of dice are thrown, find the probability that the sum of the face numbers will be 6.

As each die has 6 faces, the pair can be thrown in $6 \times 6 = 36$ different ways. The desired events are of three kinds : 1 and 5, 2 and 4, 3 and 3. Now the first die may come up 1 and the second 5, or *vice versa*, so that the pair may be 1 and 5 in two ways. Similarly, the pair may be 2 and 4 in two ways. The pair can be 3 and 3 in a single way. Hence the probability of throwing 6 is $\frac{2}{36} + \frac{2}{36} + \frac{1}{36} = \frac{5}{36}$.

EXAMPLE 5. In a lottery there are 10 blanks and 3 prize tickets of $100 each and 2 prize tickets of $10 each. What is the value of the expectation of a holder of 3 tickets ?

We form a table in which the first column indicates the nature of the event, the second column its probability, the third column the total value of the event if successful, the fourth column the value of the expectation (probable value) of the event.

Event.	Probability.	Value of Event.	Expectation.
Three $100 prizes............	1 ÷ 455	$300	$300 ÷ 455
Two $100, one $10..........	6 ÷ 455	210	1260 ÷ 455
One $100, two $10...........	3 ÷ 455	120	360 ÷ 455
Two $100, one blank.........	30 ÷ 455	200	6000 ÷ 455
Two $10, one blank..........	10 ÷ 455	20	200 ÷ 455
One $100, one $10, one blank.	60 ÷ 455	110	6600 ÷ 455
One $100, two blanks........	135 ÷ 455	100	13500 ÷ 455
One $10, two blanks.........	90 ÷ 455	10	900 ÷ 455
Three blanks................	120 ÷ 455	0	0

The sum of the numbers in the last column is 29120 ÷ 455, or 64. Hence the expectation of the holder has the value $64. As a check, the sum of the fractions in the second column is unity.

96. *If two events are independent* (the occurrence of one not contingent upon the occurrence of the other), the probability that both will happen equals the probability of the first event's happening multiplied by the probability of the second event's happening.*

Suppose that the first event can happen in m ways and fail in n ways, each of these ways being equally likely to occur. Let the corresponding numbers for the second event be M and N. We may associate any one of the $m + n$ occasions in which the first event happens or fails with any one of the $M + N$ occasions for the second event. Consider the resulting $(m + n)(M + N)$ occasions. In mM of these occasions both events happen. Hence the probability that both events will happen is

$$\frac{mM}{(m+n)(M+N)} = \text{Prob. of 1st} \times \text{Prob. of 2d.}$$

In general, if P_1, P_2, $P_3, \ldots$ are the respective probabilities of a number of independent events, $P_1 \cdot P_2 \cdot P_3 \ldots$ is the probability that all will happen.

For example, the probability of throwing an ace in the first and second, but not in the third, of three successive throws of a die is $\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6} = \frac{5}{216}$, since the probability of throwing an ace at one throw is $\frac{1}{6}$, and the probability of not throwing an ace is $\frac{5}{6}$.

EXAMPLE (Poincaré). From two urns of like exterior, the first containing 4 white and 6 black balls, the second containing 3 white and 2 black balls, one ball is drawn at random. Find the probability P that a white ball is drawn.

Since the total number of balls in the two urns is 15 and the number of white balls is 7, one might be led to say that the probability P of drawing a white ball is 7/15. But this solution is incorrect as it assumes that the events are equally probable. Indeed, the probability of drawing a particular ball from the first urn is 1/10, while the probability of drawing a given ball from the second urn is 1/5.

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* We may state the condition that A and B are independent events in the following form. The probability that A will happen, the probability that A will happen if B happens, and the probability that A will happen if B does not happen shall all three be equal. See Poincaré, Calcul des Probabilités, Paris, 1896.

The correct solution is as follows. As each urn is equally likely to be selected, the probability that the first urn will be selected is $\frac{1}{2}$, the probability that the second urn will be selected is $\frac{1}{2}$. If a ball is taken from the first urn, the probability that it will be a white ball is $\frac{4}{10}$. Applying the theorem which relates to two independent events, we find that the probability that the first urn will be selected and that a white ball will be taken from it is $\frac{1}{2} \cdot \frac{4}{10} = \frac{2}{10}$. Analogously, the probability that the second urn will be selected and that a white ball will be taken from it is $\frac{1}{2} \cdot \frac{3}{5} = \frac{3}{10}$. The required probability P is therefore $\frac{2}{10} + \frac{3}{10} = \frac{1}{2}$.

EXERCISES.

1. From a bag containing 4 white and 8 black balls, 3 balls are drawn. Find the chance (1) that all are white; (2) that 2 are white and 1 is black.

2. A has 3 shares in a lottery in which there are 3 prizes and 15 blanks; B has 2 shares in a lottery in which there are 3 prizes and 9 blanks. Compare their chances of drawing at least one prize.

3. If 6 coins are tossed, find the chance for 3 heads and 3 tails.

4. If 3 Latin, 5 Greek, and 6 French books are placed on a shelf at random, find the chance that the books of each language will be together.

5. The letters of the word *probability* are placed at random in a line. Find the chance that two vowels will come together.

6. Find the chance for a sum 4 at one throw of two dice.

7. Find the chance for a sum 15 at one throw of three dice.

8. Find the chance for a sum of at least 6 at one throw of two dice.

9. In a lottery of 100 tickets, there are 5 prizes of $100, 10 of $50, and 10 of $20. Find the value of a ticket.

10. From a bag containing 5 twenty-dollar bills and 15 ten-dollar bills, a person is entitled to draw 2 bills. What is the value of his expectation?

11. Find the probability of throwing an ace at least once in three successive throws of a single die.

12. From a bag containing 4 white, 5 black, and 6 red balls, 3 balls are drawn in succession, each being replaced prior to the next draw. Find the probability that (1) the balls are of different colors; (2) the first is white, the second black, the third red?

13. If two events are dependent (the occurrence of one being contingent upon the occurrence of the other), the probability that both will happen equals the probability of the first multiplied by the probability that when the first has happened the second will follow. Prove a similar theorem for any number of events.

14. Solve Ex. 12 when the balls are not replaced after being drawn.

15. If P is the probability of the happening of an event and therefore $1 - P$ the probability of its not happening, the probability that the event will happen exactly r times in n trials is

$$_nC_r\ P^r(1-P)^{n-r}$$

16. If 6 coins are tossed, find the chance for a single head.

17. Find the chance of throwing exactly 3 aces in 6 throws of a die.

18. Find the chance of throwing at least 3 aces in 6 throws of a die.

19. The probability that A can solve a given problem is $\frac{2}{3}$ and B's probability is $\frac{3}{4}$. Find the probability that both together will solve it.

20. A number of 3 digits is formed at random from the 10 figures 0, 1, 2, . . . , 9. Find the chance that the sum of the 3 digits is 25.

21. A's probability of being alive 20 years hence is $\frac{3}{8}$, B's probability is $\frac{4}{5}$. Find the probability that both will be alive.

22. Two balls are drawn from a bag containing 3 red, 4 white, and 5 black balls. What is the chance that both are of the same color ?

23. A has twice the skill that B has. What is the chance that A wins 6 games before B wins 3 ?

24. What is the chance that in 5 numbers taken at random exactly two begin with the digit 9 ? That at least two begin with 9 ?

CHAPTER XI.

MATHEMATICAL INDUCTION.

97. By the method of summing an arithmetical progression we find that the sum of the first n odd numbers is n^2. This result may also be proved by the following method. We observe that $1 + 3 = 2^2$, $1 + 3 + 5 = 3^2$, $1 + 3 + 5 + 7 = 4^2$. Let us suppose that we have continued this numerical verification of the theorem as far as the first m odd numbers, so that

$$1 + 3 + 5 + \dots + (2m - 1) = m^2.$$

Consider next the sum of the first $m + 1$ odd numbers. This sum may be derived from the preceding sum by adding the $m + 1$st odd number, which is $2m + 1$. Adding it also to the second member m^2, we get

$$1+3+5 + \dots + (2m-1)+(2m+1)=m^2+(2m+1) \equiv (m+1)^2.$$

In the left member of this equation we have the sum of the first $m + 1$ odd numbers; in the right member we have the square of $m + 1$. Hence the theorem holds also for $n = m + 1$ when it has been established for $n = m$. But we observed from the identity $1 + 3 = 2^2$ that the theorem holds for $n = 2$; hence by our proof it holds for $n = 2 + 1 = 3$. Being true for $n = 3$, the theorem holds for $n = 3 + 1 = 4$. Being true for $n = 4$, it holds for $n = 4 + 1 = 5$, etc. It therefore follows that the theorem holds for any positive integer n.

98. This method of proof is called **mathematical induction.** There are two distinct *parts* in the proof as applied to establish a theorem true for every positive integral value of n. On the one hand, we have to show that, if the theorem holds for any particular value m of n, it will then hold for the next value $m + 1$ of n. On the other hand, we have to show that the theorem is true for the initial values of n, say $n = 1$, or $n = 2$, so that there may be a starting point (free from "ifs") for the induction process from m to $m + 1$ (from "if true for m" to "then true for $m + 1$"). We must have a ladder by which to climb from any round (the mth) to the next round (the $m + 1$st); but the ladder must rest on a solid basis so that we can get on to the ladder (the $n = 1$ or $n = 2$ rounds).

To illustrate the necessity of *both* parts in the proof, consider the two following examples.

Whatever be the value of the number c, the equation

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} + c$$

will be true for $n = m + 1$ if true for $n = m$. Indeed, we have

$$(1 + 2 + 2^2 + \ldots + 2^m) + 2^{m+1} = (2^{m+1} + c) + 2^{m+1} = 2^{m+2} + c,$$

which is the original formula for $n = m + 1$. The "ladder" is therefore perfect. But to be able to climb up the ladder to the general round (the nth round), we must be able to get on the first round; the formula must be true for $n = 1$, which requires $1 + 2 = 2^2 + c$, whence $c = -1$. Hence the only true formula is

$$1 + 2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 1,$$

a result also proved by the formula for the sum of a geometrical progression.

As the second example, consider the theorem due to Gauss that, for prime* numbers n, the circumference of a circle can be divided into n equal parts by means of ruler and compass if, and only if, n has the form $n = 2^{2^m} + 1$. For $m = 0, 1, 2, 3, 4$, we find that $n = 3, 5, 17, 257, 65537$, respectively, and that each of these values of n is a prime number. It was consequently supposed by Fermat that the expression $2^{2^m} + 1$ was a prime number for all positive integral values of m. Later, however, Euler proved that this supposition was false by showing that, for $m = 5$, the resulting number $2^{2^5} + 1 = 2^{32} + 1 = 4,294,967,297$ is divisible by 641.

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* A positive integer n is called *prime* when it is divisible by no positive integers other than unity and n itself. Thus 2, 3, 5, 7, 11, . . . , are primes.

99. EXAMPLE 1. Prove by mathematical induction that the sum of the cubes of the first n natural numbers equals $\{\frac{1}{2}n(n+1)\}^2$.

By trial, $1^3 + 2^3 = \{\frac{1}{2}2(3)\}^2$. Suppose, then, that the theorem holds for any particular value m of n, so that

$$1^3 + 2^3 + 3^3 + \dots + m^3 = \{\tfrac{1}{2}m(m+1)\}^2.$$

Adding $(m+1)^3$ to both members, we get

$$1^3 + 2^3 + 3^3 + \dots + m^3 + (m+1)^3 = (m+1)^2 \left\{(m+1) + \frac{m^2}{4}\right\}$$

$$= (m+1)^2 \left\{\frac{m^2 + 4m + 4}{4}\right\} = \{\tfrac{1}{2}(m+1)(m+2)\}^2.$$

Hence the theorem holds also for the value $m + 1$ of n. Being true for $n = 2$, the theorem therefore holds for $n = 3$, etc. By the two parts, the theorem is proved to hold for every n.

EXAMPLE 2. Show that $x^n - y^n$ is divisible by $x - y$ if n is a positive integer.

The theorem is evidently true for $n = 1$ and $n = 2$. Since

$$x^m - y^m \equiv x^{m-1}(x - y) + y(x^{m-1} - y^{m-1}),$$

the theorem will be true for $n = m$ if true for $n = m - 1$. In fact, if $x^{m-1} - y^{m-1}$ is divisible by $x - y$, the identity shows that $x^m - y^m$ will also be divisible by $x - y$. Hence the theorem follows by induction.

EXAMPLE 3. Prove by induction the Binomial Theorem for all positive integral exponents n.

By § 91, the theorem in question is given by the formula

$$(a + b)^n = c_0a^n + c_1a^{n-1}b + c_2a^{n-2}b^2 + \dots + c_ra^{n-r}b^r + \dots + c_nb^n,$$

where

$$c_0 = 1,\ c_1 = n,\ c_2 = \frac{n(n-1)}{1 \cdot 2}, \dots,\ c_r = \frac{n(n-1)\dots(n-r+1)}{1 \cdot 2 \dots r}, \dots,\ c_n = 1.$$

[illegible] $(a+b)^1 = a+b$; also for $n=2$, since

$$[illegible] = c_0a^2 + c_1ab + c_2b^2.$$

[illegible] established for $n = m$, so that

$$[illegible] + \ldots + c_ra^{m-r}b^r + \ldots + c_mb^m$$

[illegible] we get

$$[illegible] - c_2a^{m-1}b^2 + \ldots$$
$$[illegible] + \ldots + (c_{m-1} + c_m)ab^m + c_mb^{m+1}.$$

[illegible]

$$[illegible] \ldots (m-r+1) \over 1 \cdot 2 \ldots r$$

$$[illegible] (m-r+1)\};$$

$$[illegible] + \ldots$$

$$[illegible] + \ldots + b^{m+1}.$$

[illegible] for $n=m+1$. [illegible] Also the theorem [illegible] being true for $n=1$, it [illegible]

[illegible]

[illegible] and last terms [illegible] binomial [illegible] $n > 2$ [illegible] coefficients [illegible] integer. Moreover, it is divisible by n since there is no factor in common with n and the denominator $r!$ In fact n is greater than r and hence cannot divide any factor of $r!$; while, inversely, no factor of $r!$ can divide the *prime* number n.

$$\therefore \quad f(M+1)=f(M)+\text{a multiple of } n.$$

Hence, if $f(M)$ is divisible by n, so is also $f(M+1)$. But $f(1)\equiv 0$. Hence $f(2)$ is divisible by n; therefore also $f(3)$, etc.

This theorem is known as Fermat's Theorem. Incidentally it was shown that in the expansion of $(a+b)^n$, for n a prime number, all the binomial coefficients except the first and last are divisible by n.

As illustrations of the theorem, we observe that

$$2^5-2=5\times 6,\quad 3^5-3=5\times 48,\quad 4^5-4=5\times 12\times 17.$$

EXERCISES.

Prove by mathematical induction that

1. $1^2+2^2+3^2+\dots+n^2=\frac{1}{6}n(n+1)(2n+1)$.
2. $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\dots+\frac{1}{n(n+1)}=\frac{n}{n+1}$.
3. $(a+b+c+d)^n-a^n-b^n-c^n-d^n$ is divisible by the *prime* n.
4. x^n-y^n is divisible by $x+y$ when n is even.
5. $2\cdot 4+4\cdot 6+6\cdot 8+\dots+2n(2n+2)=\frac{n}{3}(2n+2)(2n+4)$.
6. $2\cdot 7^n+3\cdot 5^n-5$ is divisible by 24.
7. $1+\frac{m}{1}+\frac{m(m+1)}{1\cdot 2}+\frac{m(m+1)(m+2)}{1\cdot 2\cdot 3}+\dots$ to $n+1$ terms equals
$$\frac{(m+1)(m+2)\dots(m+n)}{1\cdot 2\dots n}.$$
8. $1\cdot 2+2\cdot 3+3\cdot 4+\dots+n(n+1)=\frac{1}{3}n(n+1)(n+2)$.
9. $1\cdot 2\cdot 3+2\cdot 3\cdot 4+\dots+n(n+1)(n+2)=\frac{1}{4}n(n+1)(n+2)(n+3)$.
10. $1^3+2^3+\dots+n^3=(1+2+\dots+n)^2$.
11. $(1^5+2^5+\dots+n^5)+(1^7+2^7+\dots+n^7)=2(1+2+\dots+n)^4$.
12. $(1^3+2^3+\dots+n^3)+3(1^5+2^5+\dots+n^5)=4(1+2+\dots+n)^3$.

The theorem is true for $n = 1$, since $(a+b)^1 = a+b$; also for $n = 2$, since

$$(a+b)^2 = a^2 + 2ab + b^2 = c_0a^2 + c_1ab + c_2b^2.$$

Suppose that the theorem has been established for $n = m$, so that

$$(a+b)^m = c_0a^m + c_1a^{m-1}b + c_2a^{m-2}b^2 + \ldots + c_ra^{m-r}b^r + \ldots + c_mb^m$$

Multiplying both members by $a + b$, we get

$$(a+b)^{m+1} = c_0a^{m+1} + (c_0 + c_1)a^mb + (c_1 + c_2)a^{m-1}b^2 + \ldots$$
$$+ (c_{r-1} + c_r)a^{m-r+1}b^r + \ldots + (c_{m-1} + c_m)ab^m + c_mb^{m+1}.$$

But $c_0 = 1$, $c_m = 1$, and

$$c_0 + c_1 = m + 1,$$
$$c_1 + c_2 = m + \frac{m(m-1)}{1\cdot 2} = \frac{m(m+1)}{1\cdot 2},$$
$$c_{r-1} + c_r = \frac{m(m-1)\ldots(m-r+2)}{1\cdot 2\ldots(r-1)} + \frac{m(m-1)\ldots(m-r+1)}{1\cdot 2\ldots r}$$
$$= \frac{m(m-1)\ldots(m-r+2)\{r + (m-r+1)\}}{1\cdot 2\ldots(r-1)r},$$
$$= \frac{(m+1)m(m-1)\ldots(m-r+2)}{1\cdot 2\ldots(r-1)r}.$$

$$\therefore\quad (a+b)^{m+1} = a^{m+1} + (m+1)a^mb + \frac{(m+1)m}{1\cdot 2}a^{m-1}b^2 + \ldots$$
$$+ \frac{(m+1)(m)(m-1)\ldots(m+1-r+1)}{1\cdot 2\cdot 3\ldots r}a^{m+1-r}b^r + \ldots + b^{m+1}.$$

But this formula is the same as the initial formula for $(a+b)^n$ for $n = m+1$. Hence the theorem is true for $n = m+1$ if true for $n = m$. Also the theorem is true for $n = 1$. Hence it is always true. In fact, being true for $n = 1$, it is true for $n = 2$, and therefore for $n = 3$, etc.

EXAMPLE 4. If n is a prime number, $N^n - N$ is divisible by n.

Denote $N^n - N$ by the functional symbol $f(N)$. Then

$$f(M+1) - f(M) = \{(M+1)^n - (M+1)\} - \{M^n - M\}$$
$$= nM^{n-1} + \frac{n(n-1)}{1\cdot 2}M^{n-2} + \ldots + nM,$$

upon expanding $(M+1)^n$ by the Binomial Theorem. The first and last terms are evidently divisible by n. Also $\frac{n(n-1)}{1\cdot 2}$ is an integer, being a binomial coefficient, and is divisible by n, since 2 does not divide the prime n ($n > 2$, otherwise the term M^{n-2} does not occur). In general, the coefficient $\frac{n(n-1)\ldots(n-r+1)}{1\cdot 2\ldots r}$ of M^{n-r} occurs only when $n > r$ and is then an integer. Moreover, it is divisible by n since there is no factor in common with n and the denominator $r\,!$ In fact n is greater than r and hence cannot divide any factor of $r\,!$; while, inversely, no factor of $r\,!$ can divide the *prime* number n.

$$\therefore \quad f(M+1) = f(M) + \text{a multiple of } n.$$

Hence, if $f(M)$ is divisible by n, so is also $f(M+1)$. But $f(1) \equiv 0$. Hence $f(2)$ is divisible by n; therefore also $f(3)$, etc.

This theorem is known as Fermat's Theorem. Incidentally it was shown that in the expansion of $(a+b)^n$, for n a prime number, all the binomial coefficients except the first and last are divisible by n.

As illustrations of the theorem, we observe that

$$2^5 - 2 = 5 \times 6, \quad 3^5 - 3 = 5 \times 48, \quad 4^5 - 4 = 5 \times 12 \times 17.$$

EXERCISES.

Prove by mathematical induction that

1. $1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{1}{6}n(n+1)(2n+1)$.
2. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \ldots + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.
3. $(a+b+c+d)^n - a^n - b^n - c^n - d^n$ is divisible by the *prime* n.
4. $x^n - y^n$ is divisible by $x + y$ when n is even
5. $2\cdot 4 + 4\cdot 6 + 6\cdot 8 + \ldots + 2n(2n+2) = \dfrac{n}{3}(2n+2)(2n+4)$.
6. $2\cdot 7^n + 3\cdot 5^n - 5$ is divisible by 24.
7. $1 + \dfrac{m}{1} + \dfrac{m(m+1)}{1\cdot 2} + \dfrac{m(m+1)(m+2)}{1\cdot 2\cdot 3} + \ldots$ to $n+1$ terms equals
$$\frac{(m+1)(m+2)\ldots(m+n)}{1\cdot 2\ldots n}.$$
8. $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.
9. $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3)$.
10. $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$.
11. $(1^5 + 2^5 + \ldots + n^5) + (1^7 + 2^7 + \ldots + n^7) = 2(1 + 2 + \ldots + n)^4$.
12. $(1^3 + 2^3 + \ldots + n^3) + 3(1^5 + 2^5 + \ldots + n^5) = 4(1 + 2 + \ldots + n)^3$.

CHAPTER XII

LIMITS ; INDETERMINATE FORMS.

100. Preliminary Remarks on Limits.—In testing the penetrating power of a bullet fired at a given distance with a particular rifle and uniform loads, we determine the limit to the thickness of the pine board through which the bullet will pass. In Geometry, the length of the circumference of a circle is obtained as the limit of the perimeters of the inscribed (also of the circumscribed) regular polygons as the number of sides increases without bound. Of all triangles having a given perimeter and a given base, the isosceles has greatest area, so that there is a limit to the area of such a triangle. The last two illustrations offer a contrast in one respect, the variable inscribed regular polygon does not reach its limit, the circumference; but in the last example the variable triangle reaches the limiting form of greatest area, the isosceles triangle.

In the algebraic formulation, we consider a variable x and a function $f(x)$ to which there is a definite value for each value that x may take (see definition of **function,** § 75).

A Necessary Condition for a Limit.—In order that the function $f(x)$ shall have the limit l as x approaches a, where l and a are fixed quantities, it must be possible to make the difference between $f(x)$ and l as small as we please by taking x sufficiently near to a. [Complete definition of limit in §§ 103–104.]

This condition is satisfied by the function $2x + 4$ if $a = 3$, $l = 10$, since the difference between $2x + 4$ and 10 is proportional to the difference between x and 3. In this case we may let x become 3, whence $2x + 4$ becomes 10.

The question is not so simple with the function

$$f(x) = \frac{x^2 - a^2}{x - a},$$

which is not defined for $x = a$. Indeed, upon setting $x = a$, we are led to the symbol $\frac{0}{0}$, which may be regarded to be equal to any number whatever if, indeed, it have a value at all (§ 102). At any rate, $f(x)$ does not have a definite value for $x = a$ and hence, by the definition of a function, is not defined for $x = a$. We may, however, show that $f(x)$ approaches a limit as x approaches a. As long as x differs from a, $f(x)$ equals $x + a$. Hence

$$f(a + .01) = 2a + .01, \qquad f(a + .001) = 2a + .001,$$
$$f(a + .0001) = 2a + .0001, \ldots$$

By making the difference between x and a sufficiently small, but distinct from zero, the difference between $f(x)$ and $2a$ may be made as small as we please. That this necessary condition is also a sufficient condition for the limit $2a$ is shown in § 105.

101. The Term Infinity.—The value of the function $\frac{10}{x}$ for any value of x except zero is readily computed. The operation division ceases to have a meaning when the divisor is zero; there is no number q such that $0 \cdot q = 10$, since, for every number q, $0 \cdot q$ is always 0; 10 cannot be separated into parts each equal zero. But as x decreases through positive values, $\frac{10}{x}$ increases and may be made to exceed any assigned value (say 10,000) by taking x sufficiently small (viz., less than one-thousandth). Thus $\frac{10}{x}$ increases without bound; for brevity, it is said to "become **infinite**," or to "approach **infinity**" (symbol, $+\infty$, or merely ∞), terms which mean only "increases without bound." Similarly, as x increases through negative values, $\frac{10}{x}$ decreases, and as x approaches zero, $\frac{10}{x}$ decreases without bound; it is said to become **negative infinity** (symbol, $-\infty$). A variable which does not increase without bound nor decrease without bound is said to remain **finite.**

102. While the function $\frac{10x}{x}$ is not defined for $x = 0$, division by zero being excluded (§ 101), it has the limit 10 as x approaches zero. In fact, $\frac{10x}{x} = 10$ for every $x \neq 0$. But if we consider the numerator and denominator separately, we have functions which approach 0 as $x \doteq 0$, and the quotient of their limits takes the form $\frac{0}{0}$. But the same symbol would be obtained from the functions $\frac{x}{x}, \frac{3x}{x}, \frac{10x}{x}$, whereas their limits as $x \doteq 0$ are 1, 3, 10, respectively. Hence the symbol $\frac{0}{0}$, considered apart from its origin, does not possess a definite value and so is called an **indeterminate form.**

For every value of $x > 0$ the function $\frac{10x}{x}$ has the value 10, so that its limit as* $x \doteq \infty$ is 10. If we consider the numerator and denominator separately, we have functions which increase without bound as x does, and the quotient of their limits has the form $\frac{\infty}{\infty}$. But, as $x \doteq \infty$, the functions $\frac{x}{x}, \frac{3x}{x}, \frac{10x}{x}$ have the limits 1, 3, 10, respectively, while each may lead, in the sense explained, to the symbol $\frac{\infty}{\infty}$. Hence $\frac{\infty}{\infty}$ is called an **indeterminate form,** not possessing a definite value.

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* Read "x increases without bound." $x \doteq -\infty$ is read "x decreases without bound." But $x \doteq 1$ is read "as x approaches 1."

Writing the above functions in the forms $x \cdot \frac{1}{x}$, $x \cdot \frac{3}{x}$, $x \cdot \frac{10}{x}$, and considering the limits of the separate factors as $x \doteq 0$, we are led to the symbol $0 \cdot \infty$ in each case. Similarly, as $x \doteq \infty$, we are led to the symbol $\infty \cdot 0$ in each case. But $\frac{x}{x}$, $\frac{3x}{x}$, $\frac{10x}{x}$ have the respective limits 1, 3, 10 as $x \doteq 0$ or as $x \doteq \infty$. Hence the symbols $0 \cdot \infty$ and $\infty \cdot 0$, when considered apart from their origin, are indeterminate forms.

As $x \doteq 1$, the two members of the identity

$$\frac{2}{1-x^2} - \frac{1}{1-x} \equiv \frac{1-x}{1-x^2}$$

lead to the respective symbols $\infty - \infty$ and $\frac{0}{0}$. Thus $\infty - \infty$ is an indeterminate form, when considered apart from its origin.

If, for $x = a$, a function of x assumes an indeterminate form, we determine its limit as $x \doteq a$. If the given function is $f(x)/g(x)$, where $f(x) \doteq 0$ and $g(x) \doteq 0$ as $x \doteq a$, we seek $\underset{x = a}{\text{limit}} \frac{f(x)}{g(x)}$.

EXAMPLE 1. As $x \doteq 1$, $(1 - x^5)/(1 - x^3)$ leads to the symbol 0/0. But

$$\frac{1-x^5}{1-x^3} = \frac{(1-x)(1+x+x^2+x^3+x^4)}{(1-x)(1+x+x^2)},$$

$$\therefore \underset{x \doteq 1}{\text{limit}} \frac{1-x^5}{1-x^3} = \frac{5}{3}.$$

EXAMPLE 2. Find the limit as $x \doteq \infty$ of

$$f(x) = \frac{4x^2 - 3x + 7}{x^2 - 5x + 1},$$

which may be said to lead to the symbol ∞/∞.

Dividing numerator and denominator by x^2, we see that

$$\underset{x \doteq \infty}{\text{limit}} f(x) = \underset{x \doteq \infty}{\text{limit}} \frac{4 - 3/x + 7/x^2}{1 - 5/x + 1/x^2} = 4.$$

EXAMPLE 3. The limit, as x approaches 3, of

$$\frac{\sqrt{3x}-3}{x-3}=\frac{\sqrt{3}\,(\sqrt{x}-\sqrt{3})}{(\sqrt{x}+\sqrt{3})(\sqrt{x}-\sqrt{3})}$$

is $\sqrt{3}/2\sqrt{3}\equiv\frac{1}{2}$. For $x=3$, the fraction leads to the symbol 0/0.

EXERCISES.

State which indeterminate form arises for the value of x given in each of the following problems. Find the limit of the function as x approaches the value given.

1. $\dfrac{x^2-1}{x^3+1}$ for $x=-1$.

2. $\dfrac{x^2-x-2}{x^2+2x-8}$ for $x=2$.

3. $\dfrac{x^n-1}{x-1}$ for $x=1$.

4. $\dfrac{\sqrt{x^3}-\sqrt{y^3}}{\sqrt{x}-\sqrt{y}}$ for $x=y$.

5. $\dfrac{\sqrt{x}-1}{\sqrt{x-1}}$ for $x=1$.

6. $\dfrac{2}{2-x}-\dfrac{5}{2-\sqrt{2x}}$ for $x=2$.

7. $\dfrac{(x-3)(7x+1)}{x^2-5x}$ for $x=\infty$.

8. $\dfrac{x^2+7x}{x^3-3x}$ for $x=0$.

9. $\dfrac{5}{1+x}+\dfrac{2}{1-x^2}$ for $x=1$.

10. $\left(x+\dfrac{1}{2+x}\right)(x^2-x-6)$ for $x=-2$.

11. $\dfrac{3-\sqrt{x+7}}{x-2}$ for $x=2$.

12. $\dfrac{\sqrt{1+3x}-\sqrt{x-1}}{1+x}$ for $x=-1$.

103. In giving a complete definition of limit, which shall give not only a necessary condition for a limit, but the necessary and the sufficient conditions, we give in this section the definition of a limit as $n \doteq \infty$, and in § 104 the definition of a limit as $x \doteq a$, where a is a given constant. The former case is the one used in the later chapters of the text.

In giving a necessary condition for a limit (§ 100), nothing was said of the effect arising by letting x approach a by different methods, for example, by passing through positive values only, negative values only, or rational values only, or *even* integral values only, etc. Thus the sum of an even number of terms of the geometrical progression 1, − 1, + 1, − 1, + 1, . . . is zero, the sum of an odd number of terms is + 1. As the number of terms is increased without bound, there is no limit approached. If n grows large through even values only, the necessary condition of § 100 for the limit zero is satisfied, whereas we do not admit that there exists a limit of the progression.

In § 68 was discussed the sum

$$S_n \equiv 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}},$$

and it was shown that the difference between S_n and 2 may be made as small as we please by choosing n sufficiently large. Thus to make $\frac{1}{2^{n-1}} < \frac{1}{100}$, it suffices to take $n > 7$; to make $\frac{1}{2^{n-1}} < \frac{1}{1000}$, it suffices to take $n > 10$. Then S_n is said to have the limit 2 as n increases indefinitely.

Definition.* If the function $f(n)$ differs from a constant l by an amount less than an assigned positive number ϵ, however small, for all values of n which exceed an assignable number (whose value depends upon the value of ϵ), then $f(n)$ is said to have the limit l as n increases indefinitely and we write

$$\underset{n \doteq \infty}{\text{limit}} \; f(n) = l.$$

If, in the preceding example, ϵ be $\frac{1}{100}$, then $2 - S_n < \epsilon$ for $n > 7$. If ϵ be $\frac{1}{1000}$, then $2 - S_n < \epsilon$ for $n > 10$.

Consider the repeating decimal $.\dot{3} \equiv .333\dots$ and set

$$S_n \equiv .3333\dots \text{ (to } n \text{ decimal places).}$$

For $\epsilon = .001$, $\frac{1}{3} - S_3 \equiv \frac{1}{3000} < \epsilon$. For $\epsilon = .00001$, $\frac{1}{3} - S_5 \equiv \frac{1}{300000} < \epsilon$.

In general, for $\epsilon = 10^{-e}$, $\frac{1}{3} - S_n < \epsilon$ for every $n \geqq e$. Hence

$$\underset{n \doteq \infty}{\text{limit}} \; S_n = \tfrac{1}{3}.$$

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* For an infinite limit $l = \infty$, the definition would read: If $f(n)$ exceeds an assigned positive number E, however great, for all values of n which exceed an assignable number, then

$$\underset{n \doteq \infty}{\text{limit}} \; f(n) = \infty.$$

The limit of $.\dot{6} = .666\ldots$ is $\frac{2}{3}$, being twice the limit of $.\dot{3}$, a result following from the theorem :

If, as n increases indefinitely, $f(n)$ has the limit l, and if c is any constant, then $c \,.\, f(n)$ has the limit $c \,.\, l$.

In fact, for an assigned positive number ϵ', however small, $f(n)$ differs from l by an amount $< \epsilon'$ for every value of n which exceeds an assignable number. Hence $c \,.\, f(n)$ differs from $c \,.\, l$ by an amount $< c\epsilon'$ for those values of n. Taking $\epsilon' = \frac{\epsilon}{c}$, we conclude that, for an assigned positive number ϵ, $c \,.\, f(n)$ differs from $c \,.\, l$ by an amount $< \epsilon$ for every value of n which exceeds an assignable number, so that $c \,.\, f(n)$ has the limit $c \,.\, l$.

104. By way of introduction to the general definition below, a function having a limit and a function not having a limit as x approaches 3 are first given.

The limit of $2x + 4$ as x approaches 3 is 10. In fact, 10 and $2x + 4$ differ by an amount $< \epsilon$ for every x between $3 - \frac{\epsilon}{2}$ and $3 + \frac{\epsilon}{2}$.

Let a function $E(x)$ be defined for *positive* values as follows : $E(x) = x$, for x an integer; $E(x)$ equals the integer just greater than x, for x fractional or irrational. Hence as x increases from 2.1 up to 3, including 3, $E(x)$ remains equal to 3; but as x decreases from 4 toward 3, excluding 3, $E(x)$ remains equal to 4. Thus $E(x)$ does not have a limit as x approaches 3.*

General definition of limit. If the function $f(x)$ differs from a constant l by an amount less than an assigned positive number ϵ, however small, for all values of x between $a-\delta$ and $a+\delta$ (except, perhaps, $x=a$), where a is a constant and δ an assignable positive number depending upon ϵ, then $f(x)$ is said to have the limit l as x approaches a, and we write

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* In a more restricted sense of limit, as x approaches 3 through values immediately less than 3, $E(x)$ has the limit 3. Similarly, by § 101, as x approaches 0 through positive values, $\frac{10}{x}$ has the limit $+\infty$; as x approaches 0 through negative values, $\frac{10}{x}$ has the limit $-\infty$.

$$\underset{x \doteq a}{\text{limit}}\ f(x) = l.$$

For example, the limit of the above function $2x+4$ as x approaches 0 is 4, since the difference $2x$ is numerically $< \epsilon$ for every x between $-\frac{\epsilon}{2}$ and $+\frac{\epsilon}{2}$.

The definition is to be modified if either a or l is infinite or if both are infinite. Thus, if $f(x)$ exceeds an assigned positive number E, however great, for all values of x between $a-\delta$ and $a+\delta$, where a is a constant and δ is an assignable positive number, then $f(x)$ has the limit $+\infty$ as x approaches a, and we write

$$\underset{x \doteq a}{\text{limit}}\ f(x) = +\infty.$$

Similarly, the limit $-\infty$ may occur. The case $a=\infty$ is treated in § 103. For example,

$$\underset{x \doteq 0}{\text{limit}}\ \frac{1}{x^2} = \infty, \quad \underset{x \doteq \infty}{\text{limit}}\ \frac{4x-3}{2x+1} = \underset{x \doteq \infty}{\text{limit}}\ \frac{4-3/x}{2+1/x} = 2.$$

105. For all values of x different from a, the functions

$$f(x) = \frac{x^2-a^2}{x-a}, \quad F(x) = x+a$$

are equal. But $f(x)$ is not defined for $x=a$, since it then assumes the indeterminate form $\frac{0}{0}$. But $f(x)$ differs from $2a$ by less than ϵ for all values of x between $a-\epsilon$ and $a+\epsilon$. Hence

$$\underset{x \doteq a}{\text{limit}}\ f(x) = 2a = \underset{x \doteq a}{\text{limit}}\ F(x).$$

In a similar manner, we may prove the general theorem:

If two functions of x are always equal whenever each is defined and if each function has a limit as $x \doteq a$, the two limits are equal.

Suppose that two functions $f(x)$ and $g(x)$ have the respective limits F and G as x approaches a. Then for an assigned positive number $\epsilon/2$, however small, the difference between $f(x)$ and F and the difference between $g(x)$ and G are each $< \epsilon/2$ for all values of x between $a - \delta$ and $a + \delta$, where δ is an assignable positive number. Hence the difference between $f(x) + g(x)$ and $F + G$ is $< \epsilon$ for all values of x between $a - \delta$ and $a + \delta$, which proves the first of the following theorems:

The limit of the sum of two functions equals the sum of their limits. The limit of the difference equals the difference of the limits.

Denote by $|n|$ the numerical value of n, whether n is positive or negative. We may write

$$|f(x)g(x) - FG| \overset{=}{<} |f(x)g(x) - f(x)G| + |f(x)G - FG|.$$

Since $f(x)$ has the limit F, and $g(x)$ the limit G, then

$$|f(x)| - |F| < \epsilon,\ |f(x) - F| < \frac{\epsilon}{2|G|},\ |g(x) - G| < \frac{\epsilon}{2|F + \epsilon|}$$

for all values of x between $a - \delta$ and $a + \delta$, where δ is a suitably chosen positive number. Hence

$$|f(x)g(x) - FG| \overset{=}{<} \frac{|f(x)|\epsilon}{2|F+\epsilon|} + \frac{|G|\epsilon}{2|G|} < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Since the difference of $f(x)g(x)$ and FG is less than ϵ for all values of x between $a - \delta$ and $a + \delta$,

$$\underset{x \doteq a}{\text{limit}} f(x)g(x) = FG = \underset{x \doteq a}{\text{limit}} f(x) \,.\, \underset{x \doteq a}{\text{limit}} g(x).$$

We have therefore established the following theorem:

The limit as $x \doteq a$ of the product of two functions of x, each of which has a limit as $x \doteq a$, equals the product of their limits.

Similarly, we may prove the theorem:

The limit of the quotient of two functions equals the quotient of their limits, if the limit of the divisor is not zero.

CHAPTER XIII.

CONVERGENCY AND DIVERGENCY OF SERIES.

106. It was shown in Chapter VI that any term of an arithmetical progression may be derived from the first term and the common difference, also that any term of a geometrical progression may be derived from the first term and the common ratio. These progressions are examples of **series** in that they give a succession of quantities formed according to some definite law. In Chapters VIII and XI were considered other series, as $1\cdot 2 + 2\cdot 3 + 3\cdot 4 + \ldots$

If the number of terms of a series be finite, it is called a **finite series**; if the series does not terminate, it is called an **infinite series.** To form the sum of a finite series is theoretically an elementary problem and the order in which the terms are added is immaterial. In the case of an infinite series, it is necessary to define what is to be understood by the expression "sum of an infinite series." We consider the sum S_n of the first n terms of the series and let n increase without bound. For the series $1^2 + 2^2 + 3^2 + \ldots$, the sum S_n increases without bound when n does. For the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$, the sum S_n has the limit 2 when n increases without bound (§ 103). For the series $1 - 1 + 1 - 1 + \ldots$, the sum S_n is 1 or 0 according as n is odd or even, so that S_n does not approach a limit as n increases without bound.

Definitions.* An infinite series is called **convergent** if the sum

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* In problems involving infinite series, it is necessary to know whether the series is convergent or divergent. Thus, if S denote the sum of a given series, we can conclude from $a + S = b + S$ that $a = b$ if the series be convergent, but *not* if it be divergent.

S_n of its first n terms approaches a finite limit as n increases without bound, **divergent** if S_n does not approach a finite limit.

Of the preceding examples of infinite series, $1 + \frac{1}{2} + \frac{1}{4} + \ldots$ is convergent, while $1^2 + 2^2 + 3^2 + \ldots$ and $1 - 1 + 1 - 1 + \ldots$ are divergent.

By definition, a series is convergent if, and only if, $\underset{n \doteq \infty}{\text{limit}}\ S_n$ exists and is a finite number S. By § 103, the condition is that the difference between S_n and S shall be less than an assigned positive number ϵ, however small, for all integers n greater than an assignable positive integer. Hence must*

$$|S - S_n| < \epsilon$$

for every integer n greater than an assignable positive integer.

107. Consider the infinite geometrical progression $1, x, x^2, \ldots$ Then

$$S_n \equiv 1 + x + x^2 + \ldots + x^{n-1} = \frac{1}{1-x} - \frac{x^n}{1-x}.$$

If $|x| > 1$, the absolute value of x^n increases without bound as n does, so that the series is divergent. If $|x| < 1$, x^n has the limit 0 as n increases without bound (§ 69). We may show that

$$\underset{n \doteq \infty}{\text{limit}}\ S_n = \frac{1}{1-x}.$$

To take a numerical case, let $x = \frac{2}{3}$, so that $S_n = 3 - 3(\frac{2}{3})^n$. To make $3 - S_n < \epsilon$, it is necessary and sufficient to take

$$n > \frac{\log 3 - \log \epsilon}{\log 3 - \log 2}.$$

There remain the cases $x = \pm 1$. For $x = +1$, $S_n = n$ and

$$\underset{n \doteq \infty}{\text{limit}}\ S_n = \underset{n \doteq \infty}{\text{limit}}\ n = \infty.$$

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* As explained before, $|t|$ denotes the numerical value of the real number t; thus $|+2| = 2$, $|-2| = 2$. The symbol $|t|$ is read **absolute value of** t.

For $x = -1$, S_n is 1 or 0 according as n is odd or even, and hence the series $1 - 1 + 1 - 1 + \ldots$ is divergent. Note that $\frac{1}{1-x}$ is now $\frac{1}{2}$. We have therefore proved the theorem:

The infinite series $1 + x + x^2 + \ldots$ *is convergent if* $|x| < 1$, *and is divergent if* $|x| \overset{=}{>} 1$.

An infinite arithmetical progression is always a divergent series. If a be the first term and d the common difference, the nth term t_n is $a + (n-1)d$ and the sum of the first n terms is (§ 63)

$$S_n = \tfrac{1}{2}n(a + t_n) = \tfrac{1}{2}n[2a + (n-1)d].$$

$$\therefore \underset{n \doteq \infty}{\text{limit}}\, S_n = \infty.$$

Hence the series is divergent. Since $\underset{n \doteq \infty}{\text{limit}}\, t_n = \infty$, the preceding result may also be derived from the theorem:

In a convergent series, the limit of the nth term, as n increases without bound, is zero.

In proof, we note that S_n and S_{n-1} both have the limit S, so that their difference t_n has the limit zero (§ 105).

108. Comparison Test. *Of two series with all terms positive,*

$$a_1 + a_2 + a_3 + \ldots, \qquad a_1' + a_2' + a_3' + \ldots,$$

let the first be convergent. If $a_n' \overset{=}{<} a_n$, *from an assigned value of n onwards, then the second series is convergent.*

Let m be the assigned value of n, so that

$$a_n' \overset{=}{<} a_n \qquad (\text{for any } n \overset{=}{>} m).$$

If S_n' denote the sum of the first n terms of $a_1' + a_2' + \ldots$, and S_n the sum of the first n terms of $a_1 + a_2 + \ldots$, with the limit S, then

$$S_n' - S_m' \overset{=}{<} S_n - S_m < S - S_m.$$

$$\therefore S_n' < S - S_m + S_m'.$$

Hence S'_n remains finite for every value of n. But S'_n is a positive variable which increases as n increases. Hence there exists a number, say S', toward which S' approaches arbitrarily near but does not pass. Thus S'_n has the limit S'.

Comparison Test. *Of two series with all terms positive,*

$$a_1 + a_2 + a_3 + \ldots, \qquad a_1' + a_2' + a_3' + \ldots,$$

let the first be divergent. If $a'_n \geqq a_n$, from an assigned value of n onwards, the second series is divergent.

With the preceding notations, we have

$$S'_n - S'_m \geqq S_n - S_m \qquad (\text{any } n \geqq m).$$

Since S_n is always positive but does not approach a finite limit, it must increase without bound as n does. Hence

$$\underset{n \doteq \infty}{\text{limit}}\, S'_n \geqq \underset{n \doteq \infty}{\text{limit}}\, S_n + S'_m - S_m = \infty.$$

109. The simplest harmonical progression is $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ Since the nth term $\frac{1}{n}$ approaches zero as n increases without bound, so that the individual terms are ultimately very small, it might be supposed that the series would have a finite sum and hence be convergent. However, by comparing the two series

$$1 + \tfrac{1}{2} + \tfrac{1}{4} + \tfrac{1}{4} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{8} + \tfrac{1}{16} + \cdots + \tfrac{1}{16} + \ldots,$$
$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8} + \tfrac{1}{9} + \cdots + \tfrac{1}{16} + \ldots,$$

we observe that each term of the second is equal to or greater than the corresponding term of the first. If the first is divergent, the second is divergent (§ 108). But the first equals

$$1 + \tfrac{1}{2} + \tfrac{2}{4} + \tfrac{4}{8} + \tfrac{8}{16} + \cdots = 1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots$$

and is divergent. To show that there is a boundless number of terms $\frac{1}{2}$ in the last series, we observe that a term $\frac{1}{2}$ was derived from each of the parenthesized groups of

$$1 + (\tfrac{1}{2}) + (\tfrac{1}{3} + \tfrac{1}{4}) + (\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}) + (\tfrac{1}{9} + \cdots + \tfrac{1}{16}) + \ldots,$$

the final terms of the groups being $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots$, so that the number of groups is boundless. *Hence the series*

$$(1) \qquad 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} + \cdots$$

is a divergent series.

110. As a generalization of the preceding theorem, *the series*

$$(2) \qquad \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots + \frac{1}{n^p} + \cdots$$

is divergent if $p \overset{=}{<} 1$ *and convergent if* $p > 1$.

Let first $p < 1$ (including the case p negative). Then

$$\frac{1}{n^p} > \frac{1}{n} \qquad \text{(for any positive integer } n > 1\text{)}.$$

Indeed, if both terms of the inequality be multiplied by n, it becomes $n^{1-p} > 1$ and hence is true, since $1 - p$ is positive. Comparing the two series (1) and (2), we have shown that each term of (2), after the first, is greater than the corresponding term of (1), which is divergent. By § 108, series (2) is divergent.

Let next $p > 1$. Comparing (2) with the series

$$(2') \qquad 1 + \frac{1}{2^p} + \frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{4^p} + \frac{1}{8^p} + \cdots,$$

each term of (2) is equal to or less than the corresponding term of (2′), thus $\frac{1}{3^p} < \frac{1}{2^p}, \quad \frac{1}{5^p} < \frac{1}{4^p}, \ldots$ Hence if (2′) be convergent, (2) will be convergent (§ 108). But (2′) may be written

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots = 1 \div \left(1 - \frac{2}{2^p}\right),$$

being an infinite geometrical series with the common ratio $\frac{2}{2^p}$, which is less than unity since $p > 1$.

111. The series (2), including (1) as a special case, are standard series, by a comparison with which a given series may often be proved convergent or divergent, as the case may be.

Example 1. Each term of the series

$$\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\frac{5}{16}+\cdots+\frac{n+1}{n^2}+\cdots$$

is greater than the corresponding term of series (1), since

$$\frac{n+1}{n^2}=\frac{1}{n}+\frac{1}{n^2}>\frac{1}{n}.$$

Since (1) is divergent, the given series is divergent.

Example 2. The series whose nth term is

$$u_n\equiv\frac{n+3}{n^3+1}$$

is convergent. In fact, $u_n<\dfrac{n+3}{n^3}\leqq\dfrac{n+3n}{n^3}\leqq\dfrac{4}{n^2}$. Hence each term of the given series is less than the corresponding term of

$$\frac{4}{1^2}+\frac{4}{2^2}+\cdots+\frac{4}{n^2}+\cdots,$$

which is convergent, being 4 times series (2) for $p=2$.

Example 3. For positive values of x and a, the series

$$\Sigma\equiv\frac{1}{x}+\frac{1}{x+a}+\frac{1}{x+2a}+\cdots+\frac{1}{x+na}+\cdots$$

is divergent. Multiplying Σ by a and setting $y=\dfrac{x}{a}$, we get

$$a\Sigma=\frac{1}{y}+\frac{1}{y+1}+\frac{1}{y+2}+\cdots+\frac{1}{y+n}+\cdots$$

Let the integer just greater than y be m. Then

$$a\Sigma>\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+m}+\cdots,$$

the latter being only a part of the divergent series (1).

EXERCISES.

Test the series for convergency or divergency:

1. $\dfrac{1}{2}+\dfrac{1}{5}+\cdots+\dfrac{1}{n^2+1}+\cdots$ 2. $\dfrac{1}{4a}+\dfrac{1}{5a}+\cdots+\dfrac{1}{na}+\cdots$

3. $\dfrac{1}{1\cdot2}+\dfrac{1}{2\cdot3}+\cdots+\dfrac{1}{n(n+1)}+\cdots$ 4. $\dfrac{1}{1\cdot2\cdot3}+\dfrac{1}{2\cdot3\cdot4}+\cdots+\dfrac{1}{n(n+1)(n+2)}+\cdots$

5. $\frac{1}{3\cdot 4^2}+\frac{1}{4\cdot 5^2}+\frac{1}{5\cdot 6^2}+\dots$ 6. $\frac{2}{3^2\cdot 4^2}+\frac{3}{4^2\cdot 5^2}+\frac{4}{5^2\cdot 6^2}+\dots$

7. $\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\dots$ 8. $\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\dots+\frac{n+1}{n^p}+\dots$

9. $\frac{1}{xy}+\frac{1}{(x+1)(y+1)}+\frac{1}{(x+2)(y+2)}+\dots$ for x and y positive.

10. $1+2-3+1+2-3+\dots$

112. *An infinite series is convergent if the terms are alternately positive and negative, if each term is numerically less than the preceding, and if the nth term has the limit zero as* $n \doteq \infty$.

In the series $u_1 - u_2 + u_3 - u_4 + \dots$, let each u be positive and let*

$$u_1 > u_2 > u_3 > u_4 > \dots, \quad \lim_{n \doteq \infty} u_n = 0.$$

By considering the sum of an even number of terms,

$$S_{2n} \equiv (u_1 - u_2) + (u_3 - u_4) + \dots + (u_{2n-1} - u_{2n}),$$

we observe that S_{2n} is positive. Hence

$$S_{2n+1} \equiv S_{2n} + u_{2n+1}$$

is also positive. Writing S_{2n+1} in the form

$$S_{2n+1} = u_1 - (u_2 - u_3) - (u_4 - u_5) - \dots - (u_{2n} - u_{2n+1}),$$

we see that $S_{2n+1} < u_1$. Hence $S_{2n} < u_1$. Since the quantities S_2, S_4, ... S_{2n}, ... are increasing positive numbers, each $< u_1$, S_{2n} must have a finite limit l as n increases without bound. But

$$\lim_{n \doteq \infty} (S_{2n+1} - S_{2n}) = \lim_{n \doteq \infty} u_{2n+1} = 0.$$

Hence S_{2n+1} also approaches the limit l. The series $u_1 - u_2 + \dots$ is therefore convergent.

EXAMPLE. The series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$ is convergent. But

$$\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + \dots$$

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* The student should make the proof for the example $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

is divergent (see end of § 107), since

$$\lim_{n=\infty} \frac{n+1}{n} = 1.$$

For the latter series the third condition of the theorem is not satisfied, while the first and second conditions are satisfied.

113. In comparing a given series with a standard series, it is often convenient to remove a *finite* number of terms of the given series, the removal of which subtracts only a finite quantity and hence does not alter the convergency or the divergency of the given series. Thus in

$$(3) \qquad 1 + \frac{5}{1!} + \frac{5^2}{2!} + \frac{5^3}{3!} + \frac{5^4}{4!} + \frac{5^5}{5!} + \frac{5^6}{6!} + \dots$$

it is convenient to remove the first five terms, whose sum is finite. There remains the infinite series

$$\frac{5^5}{5!} + \frac{5^6}{6!} + \dots = \frac{5^5}{5!}\left(1 + \frac{5}{6} + \frac{5^2}{6 \cdot 7} + \frac{5^3}{6 \cdot 7 \cdot 8} + \dots\right)$$
$$< \frac{5^5}{5!}\left[1 + \frac{5}{6} + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^3 + \dots\right].$$

The series in brackets is an infinite geometrical progression whose sum is $\frac{1}{1 - \frac{5}{6}} = 6$. Hence the series (3) is convergent (§ 108).

114. Theorem. *An infinite series all of whose terms are positive is convergent if, after any particular term, the ratio of each term to the preceding is always less than some fixed quantity which is itself less than unity.*

The terms preceding the particular term in question may be removed without altering the convergency of the series, since their number is supposed to be finite. Let the remaining series be denoted by

$$u_1 + u_2 + u_3 + u_4 + \dots,$$

and let $$\frac{u_2}{u_1} < r, \quad \frac{u_3}{u_2} < r, \quad \frac{u_4}{u_3} < r, \dots \quad (r < 1),$$

Multiplying together the first two inequalities, the first three, etc., we find that $\frac{u_3}{u_1} < r^2$, $\frac{u_4}{u_1} < r^3$, etc. Hence

$$u_1 \overset{=}{<} u_1, \quad u_2 < ru_1, \quad u_3 < r^2u_1, \quad u_4 < r^3u_1, \ldots$$

All the quantities being positive, we find by addition that

$$u_1 + u_2 + u_3 + u_4 + \ldots < u_1(1 + r + r^2 + r^3 + \ldots).$$

The infinite geometrical progression in parenthesis has the sum $\frac{1}{1-r}$, since $r < 1$. Hence $u_1 + u_2 + \ldots + u_n$ is a positive quantity which increases as n increases, but remains $< \frac{u_1}{1-r}$, and hence has a finite limit.

But nothing follows in the case $r = 1$, since $1 + r + r^2 + \ldots + r^{n-1}$ increases without bound as n does. Although $u_1 + u_2 + \ldots + u^n$ is less than the former sum, it may have a finite limit or increase without bound.

115. Theorem. *A series with positive and negative terms is convergent if the series derived from it by making all the terms positive is convergent.*

Since the series with all terms positive has a finite sum Σ, the sum S_n of the first n terms of the given series lies between $-\Sigma$ and $+\Sigma$. Hence, if we show that S_n approaches a limit as n increases indefinitely, this limit will be finite and the series convergent. Let P_p be the sum of the p positive terms in S_n, and N_q the sum of the q negative terms in S_n after their signs are made positive. Then

$$S_n = P_p - N_q \qquad (p + q = n).$$

Also P_p and N_q are each positive and less than Σ. Hence P_p approaches a fixed value P, and N_q a fixed value N as n increases without bound, so that either p or q increases or both increase without bound. Hence

$$\underset{n \doteq \infty}{\text{limit}}\, S_n = P - N = \text{fixed finite number}.$$

116. In view of the preceding theorem, the result of § 114 leads to the more general theorem:

Ratio Test. *A series is convergent if, after any particular term, the absolute value of the ratio of each term to the preceding is always less than some fixed quantity which is less than unity.*

Thus, in the series (3) of § 113, the nth and $n+1$st terms are

$$\frac{5^{n-1}}{(n-1)!}, \quad \frac{5^n}{n!}$$

The ratio of the latter to the former is $5/n$. Hence, if $n > 5$, the ratio is at most $\frac{5}{6}$ and hence is always $< \frac{11}{12}$, for example. Hence the series is convergent. Note that here also 5 terms were removed.

For the series (1), the ratio of the nth term to the preceding term is $\frac{1}{n} \div \frac{1}{n-1} = \frac{n-1}{n} = 1 - \frac{1}{n}$. While this ratio is less than unity for every value of n, it is impossible to find a proper fraction f such that $1 - \frac{1}{n} < f$ for every n. Indeed, as $n \doteq \infty$, $1 - \frac{1}{n} \doteq 1$. To put the matter in another light, if one assigns to f the value .9999, the ratio $1 - \frac{1}{n}$ will exceed $f = .9999$ as soon as $n > 10000$. Hence the theorem fails to prove the convergency of $1 + \frac{1}{2} + \frac{1}{3} + \ldots$ By § 109, the series is divergent.

Series (3) is a special case of the series

$$(4) \qquad 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \ldots$$

The ratio of the $(n+1)$st term to the nth term is x/n and is, in absolute value, less than a quantity less than unity for all values of n greater than x. Hence, if X be the least positive integer equal to or greater than $|x|$, we need only remove the first X terms of the series, so that in the remaining series the ratio of each term to the preceding shall be in absolute value always less than a fixed quantity less than unity. *Hence series* (4) *is convergent for every x.*

117. *An infinite series all of whose terms are of like sign is divergent if, after any particular term, the ratio of each term to the preceding term is always greater than or equal to unity.*

Remove the terms preceding the particular term in question and denote the remaining infinite series by

$$u_1 + u_2 + u_3 + \dots, \; u_2 \overset{=}{>} u_1, \; u_3 \overset{=}{>} u_2, \; \dots$$

$$\therefore \; u_1 + u_2 + u_3 + \dots + u_n \overset{=}{>} u_1 + u_1 + u_1 + \dots \overset{=}{>} nu_1.$$

Since $nu_1 \doteq \infty$ as $n \doteq \infty$, $u_1 + u_2 + u_3 + \dots$ is divergent.

For example, the ratio of the nth term to the $(n-1)$st term in

$$\frac{3}{1 \cdot 2} + \frac{3^2}{2 \cdot 3} + \dots + \frac{3^{n-1}}{(n-1)n} + \frac{3^n}{n(n+1)} + \dots$$

is $3(n-1) \div (n+1)$ and hence is greater than 1 for all values of n greater than 2. Hence the series is divergent.

Ratio Test. *An infinite series of positive and negative terms is divergent if, after any particular term, the ratio of each term to the preceding term is numerically equal to or greater than a fixed quantity $r > 1$.*

Beginning with the term in question, let the series be $u_1 + u_2 + u_3 + \dots$, where, for every n,

$$\left| \frac{u_{n+1}}{u_n} \right| \overset{=}{>} r.$$

Hence $|u_{n+1}| \overset{=}{>} r^n |u_1|$, so that $\lim\limits_{n \doteq \infty} |u_{n+1}| = \infty$. But a series is divergent unless the nth term approaches zero as n increases (§ 107).

118. A more convenient form of the ratio tests of §§ 116, 117 is given in the equivalent theorem:

If in any series $u_1 + u_2 + u_3 + \dots$ there exists* a limit,

$$l = \lim_{n \doteq \infty} \left| \frac{u_{n+1}}{u_n} \right|,$$

the series is convergent if $l < 1$, divergent if $l > 1$. If $l = 1$, a further test is necessary.

As an example, consider the logarithmic series (Ch. XVI),

$$\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots - (-1)^{n-1} \frac{x^{n-1}}{n-1} - (-1)^n \frac{x^n}{n} - \dots$$

The ratio of the nth term to the $(n-1)$st term is

$$r \equiv -\frac{x(n-1)}{n} = -x\left(1 - \frac{1}{n}\right).$$

Hence $\lim |r| = |x|$, so that the series is convergent if $|x| < 1$, but divergent if $|x| > 1$. For $x = +1$, the series is convergent by § 112. For $x = -1$, the series is the divergent series $-1 - \frac{1}{2} - \frac{1}{3} - \dots$ (see § 109).

EXERCISES.

1. $\frac{1}{a} - \frac{1}{a+b} + \frac{1}{a+2b} - \frac{1}{a+3b} + \dots$ is convergent if a and b are positive.

2. $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \frac{1}{4 \cdot 5} + \dots$ is convergent.

For what values of x are the following series convergent?

3. $\frac{x}{1 \cdot 2} + \frac{x^2}{2 \cdot 3} + \frac{x^3}{3 \cdot 4} + \dots$

4. $\frac{x}{1 \cdot 2} + \frac{x^2}{3 \cdot 4} + \frac{x^3}{5 \cdot 6} + \dots$

5. $\frac{1}{2} + \frac{x}{3} + \frac{x^2}{6} + \dots + \frac{x^n}{n^2 + 2} + \dots$

6. $1 + 2x + 3x^2 + 4x^3 + \dots$

7. $\frac{1}{2} + \frac{x}{3} + \frac{x^2}{4} + \dots + \frac{x^n}{n+2} + \dots$

8. $1 + \frac{x}{4} + \dots + \frac{x^{n-1}}{n^2} + \dots$

9. $x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$

10. $1 + \frac{x}{2} + \frac{x^2}{3^2} + \frac{x^3}{4^3} + \dots$

11. $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

12. $1 + \frac{2^x}{2!} + \frac{3^x}{3!} + \frac{4^x}{4!} + \dots$

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* This limit does not exist for the series $1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \dots$

CHAPTER XIV.

POWER SERIES; EXPANSIONS INTO SERIES.

119. An infinite series of the form

(1) $$a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots,$$

in which each a is independent of x, is called a **power series in** x, since each term involves x merely in the form of a power. It may be employed in an investigation only when it is convergent. To apply the ratio test of § 118, we investigate the limit of the ratio $\frac{xa_n}{a_{n-1}}$ as n increases without bound. If the limit exists, its absolute value equals $|x| \div l$, where

$$l = \underset{n \doteq \infty}{\text{limit}} \left|\frac{a_{n-1}}{a_n}\right|.$$

Noting when $|x| \div l$ is less than 1 or greater than 1, we find that

The series (1) *is convergent if* $|x| < l$, *divergent if* $|x| > l$.

For the case $|x| = l$, a special investigation is necessary.

Corollary. *If a power series converges for* $x = \alpha$, *it converges for every value of* x *numerically less than* α.

120. Suppose that series (1) converges for $x = \alpha$, $\alpha \neq 0$. Then for $x = \alpha$ the first of the following limits exists and is finite:

$$\underset{n \doteq \infty}{\text{limit}} (a_1x + a_2x^2 + \ldots + a_nx^n) = x \underset{n \doteq \infty}{\text{limit}} (a_1 + a_2x + \ldots + a_nx^{n-1}),$$

the equality being true by § 105. Hence the series $a_1 + a_2x + \ldots$ is convergent for $x = \alpha$ and therefore, by the preceding corollary, for every x numerically less than α. In particular, it has a finite value for $x = 0$, so that the product $x(a_1 + a_2x + \dots)$ vanishes for $x = 0$. We may therefore state the theorem:

If the series $a_0 + a_1x + a_2x^2 + \dots$ *is convergent for a value of* x *different from zero, the series approaches the limit* a_0 *as* x *approaches zero.*

121. Theorem. *If the two infinite series*

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots,$$
$$b_0 + b_1x + b_2x^2 + \dots + b_nx^n + \dots$$

are convergent for $x = \alpha$, *where* α *is some quantity different from zero, and if the series are equal for every value of* x *such that* $|x| \overline{\overline{<}} |\alpha|$, then $a_0 = b_0$, $a_1 = b_1$, $\dots$, $a_n = b_n$, $\dots$, *and the two series are identical.*

By the preceding theorem, the series approach the respective limits a_0 and b_0 as x approaches 0. Hence $a_0 = b_0$. Then

$$a_1x + a_2x^2 + \dots, \quad b_1x + b_2x^2 + \dots$$

are equal and are convergent for $x \overline{\overline{<}} \alpha$. Hence

$$a_1 + a_2x + \dots, \quad b_1 + b_2x + \dots$$

are equal and are convergent for $0 < |x| \overline{\overline{<}} |\alpha|$. As before, these series approach the respective limits a_1 and b_1 as x approaches 0. Hence $a_1 = b_1$. Proceeding similarly, $a_2 = b_2$, $\dots$, $a_n = b_n$, $\dots$

The theorem is a generalization of that on rational integral functions, § 78.

Hence, under certain conditions, we may equate the coefficients of like powers of x in two power series. This principle may be applied to the solution of problems involving infinite series with undetermined coefficients.

122. The **expansion** of a function into a power series is valid only when the resulting series is convergent. For example,

$$\frac{1}{1-x}, \quad 1 + x + x^2 + x^3 + \dots \quad \textit{ad infinitum,}$$

are equal if $|x| < 1$, but not if $|x| > 1$. Thus, for $x = 2$, the fraction equals -1, while the series has an infinite sum.

EXAMPLE 1. Expand $(1-x)/(1+x^2)$ into a power series.

We seek a power series convergent for values of x such that $|x| \leqq |\alpha|$, and equal to the given fraction for those values of x. Employing undetermined coefficients, we set

$$\frac{1-x}{1+x^2} = a + bx + cx^2 + dx^3 + ex^4 + \dots$$

Multiplying both members by $1 + x^2$, we have

$$1 - x = a + bx + (a + c)x^2 + (b + d)x^3 + (c + e)x^4 + \dots$$

for all values of x for which the series is convergent. Then

$$a = 1,\ b = -1,\ a + c = 0,\ b + d = 0,\ c + e = 0, \dots$$

$$\therefore \frac{1-x}{1+x^2} = 1 - x - x^2 + x^3 + x^4 + \dots \quad \text{(when convergent).}$$

That values of x do exist for which the series is convergent follows from the more direct derivation of the series by means of the expansion

$$\frac{1-x}{1+x^2} = (1-x)\left(\frac{1}{1-(-x^2)}\right) = (1-x)(1 - x^2 + x^4 - x^6 + \dots),$$

holding for $x^2 < 1$, namely, for $-1 < x < +1$.

EXAMPLE 2. Expand $\sqrt{1-x}$ into a power series.

Assume $\sqrt{1-x} = a + bx + cx^2 + dx^3 + ex^4 + \dots$

For values of x for which the series is convergent after its terms are made positive, the square of the series may be found by the elementary rule for squaring multinomials. Hence, for such values of x,

$$1 - x = a^2 + 2abx + (b^2 + 2ac)x^2 + (2ad + 2bc)x^3 + \dots$$
$$\therefore a^2 = 1, \quad 2ab = -1, \quad b^2 + 2ac = 0, \quad 2ad + 2bc = 0, \dots$$

For the positive square root, $a = +1$. Then $b = -\frac{1}{2}$, $c = -\frac{1}{8}$, $d = -\frac{1}{16}$.

$$\therefore \sqrt{1-x} = 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} + \dots$$

In the next chapter this series is shown to be convergent if $|x| < 1$.

EXAMPLE 3. Expand $\dfrac{5x^2 + 4x + 3}{1 + x + x^2 + x^3}$ into a power series.

By the method of partial fractions (Chapter VIII), we get

$$f \equiv \frac{5x^2 + 4x + 3}{(1+x)(1+x^2)} = \frac{2}{1+x} + \frac{3x+1}{1+x^2}.$$

Then, if $x^2 < 1$, the following expansions are valid:

$$\frac{2}{1+x} = 2\{1 - x + x^2 - x^3 + \ldots + (-1)^n x^n + \ldots\},$$

$$\frac{3x+1}{1+x^2} = (3x+1)\{1 - x^2 + x^4 - \ldots + (-1)^r x^{2r} + \ldots\}.$$

$$\therefore f = 3 + x + x^2 - 5x^3 + \ldots + \{2 + (-1)^n\} x^{2n} + \{-2 + 3(-1)^n\} x^{2n+1} + \ldots$$

123. The **reversion** of a series $y = a_0 + a_1 x + a_2 x^2 + \ldots$ consists in expressing x as a series in ascending powers of y, $x = b_0 + b_1 y + b_2 y^2 + \ldots$ The method is applied only to convergent series and the solution is valid only when the resulting series is convergent. If $a_0 = 0$, then $y = 0$ if $x = 0$ (§ 120), so that $b_0 = 0$. If $a_0 \neq 0$, set $y - a_0 = z$; after reversion,

$$x = c_1 z + c_2 z^2 + c_3 z^3 + \ldots \equiv c_1(y - a_0) + c_2(y - a_0)^2 + \ldots$$

EXAMPLE 1. Effect the reversion of $y = x - 2x^2 + 3x^3 - 4x^4 + \ldots$

Assume $x = ay + by^2 + cy^3 + dy^4 + \ldots$ Substituting the value of y, we get

$$x = a(x - 2x^2 + 3x^3 - \ldots) + b(x - 2x^2 + \ldots)^2 + c(x - 2x^2 + \ldots)^3 + d(x - \ldots)^4 + \ldots$$
$$= ax + (b - 2a)x^2 + (c - 4b + 3a)x^3 + (d - 6c + 10b - 4a)x^4 + \ldots$$

We find in succession that $a = 1$, $b = 2$, $c = 5$, $d = 14$. Hence

$$x = y + 2y^2 + 5y^3 + 14y^4 + \ldots$$

EXAMPLE 2. Find a series in y that will give a solution of $x^3 - 2x + 3 = y$.

Setting $y - 3 = z$, we have $z = x^3 - 2x$. Hence z changes in sign when x does. Hence the required series for x contains no even powers of z. Set

$$x = az + bz^3 + cz^5 + dz^7 + \ldots$$

$$\therefore \quad x = a(-2x + x^3) + b(-2x + x^3)^3 + c(-2x + x^3)^5 + d(-2x + x^3)^7 + \ldots$$
$$= -2ax + (a - 8b)x^3 + (12b - 32c)x^5 + (-6b + 80c - 128d)x^7 + \ldots$$

$$\therefore \quad a = -\frac{1}{2}, \quad b = \frac{-1}{16}, \quad c = \frac{-3}{128}, \quad d = \frac{-3}{256}, \ldots$$

$$\therefore \quad x = -\tfrac{1}{2}(y - 3) - \tfrac{1}{16}(y - 3)^3 - \tfrac{3}{128}(y - 3)^5 - \tfrac{3}{256}(y - 3)^7 + \ldots$$

The solution x is valid only for values of y for which the series is convergent.

The fact that the coefficients of the even powers of z are zero may also be verified by computing them directly, as was done for the odd powers.

EXERCISES.

Expand to five terms in ascending powers of x:

1. $\frac{2+x}{(1+x^2)^2}$. **2.** $\sqrt{1+x^2}$. **3.** $\frac{2+3x}{1-x+x^2}$. **4.** $\frac{1}{2x^2-x^3}$.

5. $\frac{7+x}{1+x+x^2}$. **6.** $\frac{1+x}{1+4x+3x^2}$. **7.** $(1-x)^{\frac{2}{3}}$. **8.** $(1+x)^{-\frac{1}{3}}$.

Find the coefficient of the general term in the series for

9. $\frac{2x}{(1-x)(1+x^3)}$. **10.** $\frac{1-6x}{1+3x+2x^2}$. **11.** $\frac{x+4}{x+x^4}$. **12.** $\frac{1}{x^2+x}$.

Obtain the reversion of each of the series

13. $y = x - x^2 + x^3 - x^4 + \dots$ **14.** $y = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \dots$

Solve by series the equations

15. $y = x - x^3$. **16.** $y = 1 + 2x^2 + 4x^4 + 6x^6$. **17.** $y = 1 + x + x^3$.

CHAPTER XV.

BINOMIAL THEOREM FOR ANY INDEX.*

124. Consider the power series in x

$$(1)\ 1+nx+\frac{n(n-1)}{1\cdot 2}x^2+\ldots+\frac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot 2\cdot 3\ldots r}x^r+\ldots$$

If n is a positive integer, the series is a finite series terminating with the $n+1$st term, since the $(n+2)$nd and all subsequent terms contain the factor $n-n$. In this case the series is the expansion of the binomial $(1+x)^n$ [see the proofs in §§ 91 and 99].

When n is not a positive integer, the series is an infinite series, since no one of the factors $n-1$, $n-2$, ... can now be zero. The first step in the study of the series is consequently to determine for what values of x, if any, the infinite series is convergent. We employ the ratio tests of §§ 116 and 117. The $(r+1)$st term u_{r+1} is exhibited in the series (1). The rth term is consequently

$$u_r \equiv \frac{n(n-1)(n-2)\ldots(n-r)}{1\cdot 2\cdot 3\ldots(r-1)}x^{r-1}.$$

$$\therefore\quad \frac{u_{r+1}}{u_r}=x\left(\frac{n-r+1}{r}\right)=-x\left(1-\frac{n+1}{r}\right).$$

Hence this ratio approaches $-x$ as r increases without bound. Its absolute value will ultimately become greater than unity if $|x|>1$, so that the series is divergent in that case (§ 117). But for $|x|<1$, the absolute value of the ratio will be less than some fixed quantity which is itself less than unity for all values of r which exceed a certain number and therefore the series is convergent (§ 116). For example, if $n = 6\frac{1}{3}$, $x = -\frac{1}{2}$, the ratio will be a positive fraction for $r \overset{=}{>} 3$. [It may be shown* that the series is convergent for $x = 1$ provided $n < -1$, and for $x = -1$ provided $n > 0$.]

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*The theorem was discovered by Sir Isaac Newton. The proof given is a modification of that due to Euler.

The series (1) *is convergent if x is numerically less than unity, but is divergent if x is numerically greater than unity.*

125. Suppose that x has a fixed value which is numerically less than unity. The series (1) has a finite sum whose value depends upon n; we denote the sum by $f(n)$. Then the series

$$(2)\quad 1+mx+\frac{m(m-1)}{1\cdot 2}x^2+\ldots+\frac{m(m-1)\ldots(m-r+1)}{1\cdot 2\ldots r}x^r+\ldots$$

will have the finite sum $f(m)$. If the series (1) and (2) be multiplied together and the product be arranged according to ascending powers of x, the coefficient of x will evidently be $n+m$ and the coefficient of x^2 will be

$$\frac{n(n-1)}{1\cdot 2}+nm+\frac{m(m-1)}{1\cdot 2}=\frac{(n+m)(n+m-1)}{1\cdot 2}.$$

Likewise, the coefficient of x^3 is seen to equal

$$\frac{(n+m)(n+m-1)(n+m-2)}{1\cdot 2\cdot 3}.$$

As far as the first four terms, the product-series is of the form (1) when n is replaced by $n+m$, giving series (4). We proceed to prove that this result holds true for all the terms of the product-series. This is readily proved for the case in which n and m are any positive integers. For, in this case, the series (1) is the expansion of $(1+x)^n$ and series (2) is the expansion of $(1+x)^m$, so that the product is $(1+x)^{n+m}$, whose expansion is the series (4), since $n+m$ is a positive integer. Hence, if n and m are positive integers,

$$(3)\qquad f(n)\times f(m)=f(n+m).$$

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* Charles Smith, *Treatise on Algebra*, Art. 338.

Since n and m are *arbitrary* positive integers, they must be represented by *general letters,* the restriction to positive integers being supplementary and not expressed in the *notation* for n and m. On account of the generality of the notation for n and m, the way in which the coefficients of the series (1) and (2) combine to give the coefficients of the product-series must have been a *formal* process independent of the supplementary restriction that m and n are positive integers. The argument is based upon a principle known as "the permanence of form." Its validity in the present case is established in § 126. Hence the *form* of the coefficients of the product-series remains the same when that restriction is removed, so that the product-series is

$$(4)\quad f(n+m) = 1 + (n+m)x + \frac{(n+m)(n+m-1)}{1\cdot 2}x^2 + \dots$$

whatever the values of n and m may be. Since $|x| < 1$, the latter series is convergent. Formula (3) therefore expresses the relation between the finite sums of the three series (1), (2), and (4).

126. To give an explicit proof of formula (3), consider the coefficient of x^r in the product-series, r being a fixed positive integer. It is clearly the sum Σ of $r+1$ terms, one of which is the coefficient of x^r in series (1), and another the coefficient of x^r in series (2). We wish to prove that this sum Σ equals

$$f \equiv \frac{(n+m)(n+m-1)\dots(n+m-r+1)}{1\cdot 2\dots r}.$$

Multiplying both Σ and f by $r!$, the equation to be proved is*

$$\begin{aligned} n(n-1)\dots(n-r+1) + rmn(n-1)\dots(n-r+2)& \\ + \dots + m(m-1)\dots(m-r+1)& \\ = (n+m)(n+m-1)\dots(n+m-r+1),& \end{aligned}$$

---

* Denoting for brevity the product $p(p-1)(p-2)\dots(p-s+1)$ by p^s the relation is

$$n_r + {}_rC_1\, n_{r-1} m_1 + {}_rC_2\, n_{r-2} m_2 + \dots + {}_rC_{r-1}\, n_1 m_{r-1} + m_r = (n+m)_r$$

and is known as Vandermonde's Theorem. Compare Ex. 14, p. 99.

in each term of which n and m enter to the degree r. Now the relation has been established for the case in which n and m are both positive integers. Let n be an arbitrary quantity and let the relation be expanded and the terms arranged according to descending powers of n. We obtain an equation of the rth degree in n. Let m be any particular positive integer. Then the equation is satisfied by an infinite number of values of n, namely, the positive integers. Hence (§ 78) the equation is an identity in n and is satisfied by every value of n. Hence the relation is established for n arbitrary and m any positive integer.

Consider the original relation for any particular value of n, whether or not a positive integer. We expand the members of the relation and arrange the terms according to the powers of m, obtaining an equation of the rth degree in m. By the previous case, this equation is satisfied by all positive integers m, whose number exceeds r, and is consequently satisfied by every value of m. Hence the relation is true for n and m both arbitrary quantities.

It follows that $\Sigma = f$ and hence that the product-series is identical with the series (4).

127. By repeated application of (3), we find that

$$f(n) \times f(m) \times f(p) \times \dots \times f(q) = f(n + m + p + \dots + q).$$

Let there be s terms $n, m, p, \dots, q$ and take each equal to the fraction r/s, where r and s are positive integers. Then

$$\left\{f\left(\frac{r}{s}\right)\right\}^s = f\left(\frac{r}{s} + \frac{r}{s} + \dots + \frac{r}{s}\right) = f\left(\frac{r}{s} \times s\right) = f(r).$$

Since r is a positive integer, $f(r) = (1 + x)^r$. Hence

$$\left\{f\left(\frac{r}{s}\right)\right\}^s = (1 + x)^r; \quad f\left(\frac{r}{s}\right) = (1 + x)^{\frac{r}{s}}.$$

Hence, if n is a positive fraction $\frac{r}{s}$, series (1) *equals* $(1 + x)^{\frac{r}{s}}$.

In the identity (3) put $n = -m$. Then

$$f(m) \times f(-m) = f(0) = 1.$$

$$\therefore f(-m) = \frac{1}{f(m)} = \frac{1}{(1+x)^m} = (1+x)^{-m},$$

if m is a positive fraction. *Hence if n is a negative fraction, series* (1) *equals* $(1+x)^n$. We may now state the final theorem:

If x is in absolute value less than unity, and if n is any rational number, we have the expansion

$$(5)\quad (1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \ldots$$

This result is known as the **Binomial Theorem.**

Since any real number may be approximated to any desired degree of approximation by a rational number, we can establish, by a limiting process (§ 16), the Binomial Theorem for any real index n.

128. EXAMPLE 1. Expand $(1+x)^{-1}$ by the binomial theorem.

If $|x| < 1$, formula (5) gives, for $n = -1$, the result

$$(1+x)^{-1} = 1 + (-1)x + \frac{(-1)(-2)}{1\cdot 2}x^2 + \ldots + \frac{(-1)(-2)\ldots(-r)}{1\cdot 2\ldots r}x^r + \ldots$$

$$= 1 - x + x^2 - x^3 + \ldots + (-1)^r x^r + \ldots$$

The resulting series is an infinite geometrical progression with the ratio $-x$. Hence (§ 69) its sum is $\frac{1}{1+x}$ when $|x| < 1$.

EXAMPLE 2. Expand $\sqrt{1-y}$ by the binomial theorem.

Setting $-y = x$, $n = \frac{1}{2}$, formula (5) gives, if $|y| < 1$,

$$(1-y)^{\frac{1}{2}} = 1 + \tfrac{1}{2}(-y) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1\cdot 2}(-y)^2 + \ldots + \frac{\frac{1}{2}(\frac{1}{2}-1)\ldots(\frac{1}{2}-r+1)}{1\cdot 2\ldots r}(-y)^r + \ldots$$

$$\therefore \sqrt{1-y} = 1 - \tfrac{1}{2}y - \tfrac{1}{8}y^2 - \ldots - \frac{1\cdot 3\cdot 5\ldots(2r-3)}{r!\,2^r}y^r - \ldots$$

This result agrees with that obtained by undetermined coefficients (§ 122).

EXAMPLE 3. Find $\sqrt{99}$ to six decimal places.

$$\sqrt{99} = \sqrt{100-1} = 10\sqrt{1-\frac{1}{100}} = 10\left(1 - \frac{.01}{2} - \frac{.0001}{8} - \frac{.000001}{16} - \ldots\right),$$

upon setting $y = .01$ in the formula of Ex. 2. $\therefore \sqrt{99} = 9.949874 +$. The terms beyond the fourth do not affect the first eight decimal places.

EXAMPLE 4. Find $\sqrt[3]{66}$ to six decimal places.

$$\sqrt[3]{66} = (64+2)^{\frac{1}{3}} = 4\left(1+\frac{1}{32}\right)^{\frac{1}{3}} = 4\left[1+\frac{1}{3}\cdot\frac{1}{32}-\frac{1}{9}\left(\frac{1}{32}\right)^2+\frac{5}{81}\left(\frac{1}{32}\right)^3-\dots\right]$$
$$= 4(1+.0104166-.0001085+.0000019-\dots) = 4.0412400.$$

EXERCISES.

Expand to four terms and give the $(r+1)$st term:

1. $(1+x)^{-\frac{1}{2}}$. **2.** $(1-x)^{-3}$. **3.** $(1-3x)^{\frac{1}{3}}$. **4.** $(2-3x)^{\frac{2}{3}}$.

5. $\sqrt{1-2x^2}$. **6.** $\dfrac{1}{\sqrt[3]{1+x}}$. **7.** $x^2(1-x^3)^7$. **8.** $\sqrt[3]{x^3-x^6}$.

Find to five decimal places the value of

9. $\sqrt{47}$. **10.** $\sqrt[3]{126}$. **11.** $\sqrt[3]{1002}$. **12.** $\sqrt[5]{3128}$.

13. Find the coefficients of x^3 and x^5 in $(1+2x+3x^2)^{-2}$.

14. Find the coefficient of x^r in the expansion of $\dfrac{2-x^2}{(1-x)^3}$.

15. Find the coefficients of x^{3m}, x^{3m-1}, x^{3m+1} in $(1+x+x^2)^{-1}$.

16. The coefficient of x^4 in $(1-x+x^2-x^3+x^4)^{-3}$ is zero.

17. The coefficients of x^3 and x^7 in $(1+x+x^2+x^3)^{-2}$ are zero.

18. Prove that $(1-x)^{-2} = 1+2x+3x^2+\dots+(r+1)x^r+\dots$

CHAPTER XVI.

EXPONENTIAL AND LOGARITHMIC SERIES.

129. If $n > 1$, we have, by the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + \frac{nx}{n} + \frac{nx(nx-1)}{2!\,n^2} + \frac{nx(nx-1)(nx-2)}{3!\,n^3} + \ldots$$

$$= 1 + x + \frac{x(x-1/n)}{2!} + \frac{x(x-1/n)(x-2/n)}{3!} + \ldots$$

For the case $x = 1$, we have

$$\left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{(1-1/n)}{2!} + \frac{(1-1/n)(1-2/n)}{3!} + \ldots$$

$$\therefore\ 1 + x + \frac{x(x-1/n)}{2!} + \frac{x(x-1/n)(x-2/n)}{3!} + \ldots$$

$$= \left\{1 + 1 + \frac{(1-1/n)}{2!} + \ldots\right\}^x,$$

a result true for any value of $n > 1$. Taking the limit as n increases without bound, we get

$$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots\right)^x.$$

The general term of the series on the left is

$$\frac{x^r}{r!} = \lim_{n \doteq \infty} \frac{x(x-1/n)(x-2/n)\ldots\{x-(r-1)/n\}}{1\cdot 2\cdot 3\ldots r}.$$

For small values of r this evaluation of the limit is evident. To prove that the result holds true when r becomes large, we denote by u_r the fraction whose limit we seek. Then

$$u_r = u_{r-1}\frac{\{x-(r-1)/n\}}{r} = u_{r-1}\left(\frac{x}{r} - \frac{1}{n} + \frac{1}{rn}\right).$$

$$\therefore \lim_{n \doteq \infty} u_r = \frac{x}{r} \lim_{n \doteq \infty} u_{r-1}.$$

But the limit of u_2 was seen to be $\frac{x^2}{2!}$; hence the limit of u_3 is $\frac{x^3}{3!}$. By induction, it follows that the limit of u_r is $x^r/r!$. Set

$$(1) \qquad e \equiv 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{r!} + \cdots$$

The above result may now be written

$$(2) \qquad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^r}{r!} + \cdots$$

To evaluate e, notice that $\frac{1}{4!}$ is one-fourth of $\frac{1}{3!}$, etc. Hence

$$e = 2 + .5 + .1666 + .0417 + .0083 + .0014 + .0002 + \ldots$$

Hence $e = 2.7182$ to four decimal places. To ten places, we get

$$(3) \qquad e = 2.7182818284.$$

We may now expand a^y into a power series in y. Since

$$a = e^{\log_e a}, \quad a^y = e^{y \log_e a},$$

by the third law of indices. Putting $x = y \log_e a$, series (2) gives

$$(4) \quad a^y = 1 + y \log_e a + \frac{1}{2!}y^2(\log_e a)^2 + \frac{1}{3!}y^3(\log_e a)^3 + \cdots$$

This result is known as the **Exponential Theorem.** The relation (2) is valid for any x, and (4) for any y.

130. We may give a second proof of the result (2), following the method employed for the Binomial Theorem (Chapter XV). Set

$$F(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^r}{r!} + \dots,$$

$$F(z) = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^s}{s!} + \dots,$$

$$F(x+z) = 1 + (x+z) + \frac{(x+z)^2}{2!} + \dots + \frac{(x+z)^t}{t!} + \dots$$

For every value of x and z, these series are convergent (§ 116). Form the product of the series $F(x)$ and $F(z)$. The coefficient of $x^r z^s$ in $F(x) \cdot F(z)$ is $1/(r!s!)$. But $x^r z^s$ occurs in the series $F(x+z)$ only in the expansion of the term $(x+z)^{r+s} \div (r+s)!$ and occurs there with the coefficient (as shown by the Binomial Theorem for positive integral index)

$$\frac{1}{(r+s)!} \cdot \frac{(r+s)!}{r!\,s!} = \frac{1}{r!\,s!}.$$

Hence the series obtained as the product of the series $F(x)$ and $F(z)$ is identical with the series $F(x+z)$.

$$\therefore\ F(x) \cdot F(z) = F(x+z),$$

for all values of x and z. It follows that

$$F(x_1) \cdot F(x_2) \cdot F(x_3) \dots F(x_n) = F(x_1 + x_2 + x_3 + \dots + x_n),$$

for any positive integer n and any values of $x_1, x_2, \dots, x_n$. Taking the particular values $x_1 = x_2 = \dots = x_n = 1$, we get

$$\{F(1)\}^n = F(n).$$

But $F(1) = 1 + 1 + \frac{1}{2!} + \dots = e$. Hence $F(n) = e^n$, so that (2) is proved for the case $x = n$, a positive integer. Taking next the particular values $x_1 = x_2 = \dots = x_n = \frac{m}{n}$, where m also is a positive integer, we get

$$\left\{ F\left(\frac{m}{n}\right) \right\}^n = F\left(\frac{m}{n} + \frac{m}{n} + \dots \text{ to } n \text{ terms}\right) = F(m).$$

By the case just established, $F(m) = e^m$. Hence

$$F\left(\frac{m}{n}\right) = \sqrt[n]{e^m} = e^{\frac{m}{n}}.$$

Hence formula (2) is proved for the case $x = \frac{m}{n}$, a positive fraction. To extend the proof to the case $x = -f$, f being a positive fraction, we note that $F(-f) \cdot F(f) = F(0) = 1$; whence

$$F(-f) = \frac{1}{F(f)} = \frac{1}{e^f} = e^{-f}.$$

Hence formula (2) holds for all rational values of x. By the method of limits (§§ 16, 105), the result may be extended to any real values of x.

131. EXAMPLE. Find the sum of $1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots$ *ad infinitum.*

For $x = +1$ and $x = -1$, formula (2) gives

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots, \qquad e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$$

$$\therefore \; e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \frac{2}{6!} + \dots$$

The required sum is therefore $\frac{1}{2}(e + e^{-1})$.

EXERCISES.

1. Show that $\frac{1}{2}e^{-1} = \frac{1}{3!} + \frac{2}{5!} + \frac{3}{7!} + \frac{4}{9!} + \dots$

2. Evaluate $a^2 - b^2 + \frac{1}{2!}(a^4 - b^4) + \frac{1}{3!}(a^6 - b^6) + \dots$

3. Expand $(e^{2x} + e^{6x}) \div e^{4x}$ into a power series in x.

4. Prove that $5e = \frac{1^3}{1!} + \frac{2^3}{2!} + \frac{3^3}{3!} + \frac{4^3}{4!} + \dots$

5. Prove that $1 = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \frac{4}{5!} + \ldots$

6. Prove that $e = 1 + \frac{1^2}{2!} + \frac{2^3}{3!} + \frac{3^4}{4!} + \ldots$

7. Find the sum of $1 + \frac{2x}{1!} + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \ldots$

132. Theorem. *If* $|x| < 1$, $\log_e (1 + x)$ *equals the sum of the series*

$$(5) \qquad x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots - (-1)^r \frac{x^r}{r} + \ldots$$

Substituting $1 + x$ for a in the equation (4), we get

$$(1 + x)^y = 1 + y \log_e (1 + x) + \frac{1}{2!} y^2 \{\log_e (1 + x)\}^2 + \ldots$$

Also by the Binomial Theorem, we have, when $|x| < 1$,

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{2!} x^2 + \ldots + \frac{y(y-1)\ldots(y-r+1)}{r!} x^r + \ldots$$

In the second member the coefficient of y is

$$x + \frac{(-1)}{2!} x^2 + \ldots + \frac{(-1)(-2)\ldots(-r+1)}{r!} x^r + \ldots,$$

which, upon simplification, is series (5). The coefficient of y in the series $1 + y \log_e (1 + x) + \ldots$ is $\log_e (1 + x)$. Since the two series are equal for all values of y, we may equate the coefficients of like powers of y (§ 121), so that the theorem follows.

By equating the coefficients of y^2, we find, similarly, that

$$\tfrac{1}{2}\{\log_e (1 + x)\}^2 = \tfrac{1}{2}x^2 - \tfrac{1}{3}(1 + \tfrac{1}{2})x^3 + \tfrac{1}{4}(1 + \tfrac{1}{2} + \tfrac{1}{3})x^4 - \ldots$$

133. Replacing x by $-x$, we obtain from (5), for $|x| < 1$,

$$(6) \qquad \log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots - \frac{x^r}{r} - \ldots$$

Subtracting from (5) and using $\log \frac{a}{b} = \log a - \log b$, we get

$$(7) \qquad \log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \ldots\right).$$

Substituting in (7),

$$x = \frac{m-n}{m+n}, \text{ so that } \frac{1+x}{1-x} = \frac{m}{n},$$

we get, for any positive values of m and n,

$$(8)\ \log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \frac{1}{7}\left(\frac{m-n}{m+n}\right)^7 + \ldots\right\}.$$

134. *Table of logarithms to the base* $e = 2.71828+$; **natural logarithms.** For $m = 2$, $n = 1$, formula (8) gives

$$\log_e 2 = 2\{\tfrac{1}{3} + \tfrac{1}{3}\cdot(\tfrac{1}{3})^3 + \tfrac{1}{5}(\tfrac{1}{3})^5 + \tfrac{1}{7}(\tfrac{1}{3})^7 + \ldots\},$$

from which we get $\log_e 2 = .693147$ to six decimal places.

For $m = 3$, $n = 2$, formula (8) gives

$$\begin{aligned}\log_e 3 - \log_e 2 &= 2\,\{\tfrac{1}{5} + \tfrac{1}{3}(\tfrac{1}{5})^3 + \tfrac{1}{5}(\tfrac{1}{5})^5 + \tfrac{1}{7}(\tfrac{1}{5})^7 + \ldots\} \\ &= 2\{.2 + .002667 + .000064 + .000002\} \\ &= .405466 \quad \text{(to six decimal places).}\end{aligned}$$

By carrying the work to more places, the sixth place is seen to be 5 instead of 6. Adding $\log_e 2$, we deduce $\log_e 3 = 1.098612$.

For $m = 5$, $n = 3$, formula (8) gives $\log_e 5 = 1.609438$.

We may now deduce $\log_e 4 = 2 \log_e 2$, $\log_e 6 = \log_e 2 + \log_e 3$, $\log_e 8 = 3 \log_e 2$, $\log_e 9 = 2 \log_e 3$, $\log_e 10 = \log_e 2 + \log_e 5$. We get $\log_e 7$ from (8). We thus get the numbers in the second column of the following table:

N	$\log_e N$	$\log_{10} N$
1	0	0
2	0.693147	.301030
3	1.098612	.477121
4	1.386294	.602060
5	1.609438	.698970
6	1.791759	.778151
7	1.945910	.845098
8	2.079442	.903090
9	2.197225	.954243
10	2.302585	1.000000

To obtain $\log_{10} N$, we multiply $\log_e N$ by [see § 19]

$$\log_{10} e = \frac{1}{\log_e 10} = \frac{1}{2.302585} = .4342945.$$

Hence the numbers in the third column of the table are derived from the corresponding numbers in the second column by multiplication by the constant .4342945, which is called the **modulus of common logarithms** (namely, logarithms to the base 10). Logarithms to the base e are called **natural logarithms** or **Napierian logarithms** (in honor of their discoverer, Lord Napier).

135. Interpolation. For $m = n + a$ and $m = n + b$, formula (8) gives

$$\log_e (n+a) - \log_e n = 2 \left\{ \frac{a}{2n+a} + \frac{1}{3}\left(\frac{a}{2n+a}\right)^3 + \frac{1}{5}\left(\frac{a}{2n+a}\right)^5 + \ldots \right\},$$

$$\log_e (n+b) - \log_e n = 2 \left\{ \frac{b}{2n+b} + \frac{1}{3}\left(\frac{b}{2n+b}\right)^3 + \frac{1}{5}\left(\frac{b}{2n+b}\right)^5 + \ldots \right\}:$$

If a is relatively small compared with n, the 3rd, 5th, and higher powers of $a \div (2n + a)$ may be neglected for approximate results. Similarly for b. Then, approximately,

(9) $\{\log (n + a) - \log n\} : \{\log (n+b) - \log n\} = a : b,$

since $\dfrac{a}{2n + a} \div \dfrac{b}{2n + b} = \dfrac{a}{b} \cdot \dfrac{2n + b}{2n + a}$ and the last factor differ from 1 by $\dfrac{a - b}{2n + a}$. If (9) holds for logarithms base e, it will also hold for common logarithms base 10.

To consider a numerical example, let the base be 10 and let $n = 4550$, $a = 2$, $b = 10$. Then $\frac{a}{2n + a} = \frac{2}{9102} = .00022$ to five decimal places; $\frac{b}{2n + b} = \frac{1}{911} = .00110$ to five decimal places. Their cubes may be safely neglected. By a Table of Logarithms,

$$\log 4550 = 3.65801, \quad \log 4552 = 3.65820, \quad \log 4560 = 3.65896.$$

$$\therefore \quad \log 4552 - \log 4550 = .00019, \; \log 4560 - \log 4550 = .00095.$$

Hence by the formulæ or by the Table, the differences of the logarithms are approximately proportional to $a : b$, viz., $1 : 5$, so that (9) is true. To pass from the results given by the formulæ (base e) to those given by the Table (base 10), we must employ as multiplier the modulus, practically .4343 in value. Multiplying $\log_e 4552 - \log_e 4550$, of value 2 (.00022), by .4343, we get $\log_{10} 4552 - \log_{10} 450$, of value .00019.

For relatively small differences in the numbers, the logarithm's increase is approximately in proportion to the number's increase.

136. EXAMPLE 1. Expand $\log_e(1 - x + x^2)$ into a power series.

$$\log(1 + x + x^2) = \log\frac{1 + x^3}{1 + x} = \log(1 + x^3) - \log(1 + x)$$

$$= \left(x^3 - \frac{x^6}{2} + \quad - (-1)^r\frac{x^{3r}}{r} + \ldots\right) - \left(x - \frac{x^2}{2} + \ldots - (-1)^r\frac{x^r}{r} + \ldots\right).$$

The coefficient of x^n is $\frac{(-1)^n}{n}$ if n is not divisible by 3, but is $\frac{-2}{n}(-1)^n$ if n is divisible by 3.

EXAMPLE 2. If α, β are the roots of $ax^2 + bx + c = 0$, then

$$\log(a - bx + cx^2) = \log a + \frac{\alpha + \beta}{1}x - \frac{\alpha^2 + \beta^2}{2}x^2 + \frac{\alpha^3 + \beta^3}{3}x^3 - \quad .$$

By § 32, we have $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$. Hence

$$a - bx + cx^2 = a(1 + \alpha x)(1 + \beta x)$$

$$\therefore \quad \log(a - bx + cx^2) = \log a + (\alpha x - \tfrac{1}{2}\alpha^2x^2 + \ldots) + (\beta x - \tfrac{1}{2}\beta^2x^2 + \ldots).$$

Example 3. Prove that e is incommensurable.

Suppose that $e = \frac{m}{n}$, m and n being positive integers. Then

$$\frac{m}{n} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} + \frac{1}{(n+1)!} +$$

Multiply both members by $n!$

$$\therefore \quad m(n-1)! = \text{integer} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \ldots$$

The series beginning with $1/(n+1)$ is less than

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots = \frac{1}{n}.$$

Hence the sum of the series lies between $\frac{1}{n+1}$ and $\frac{1}{n}$ and is therefore a proper fraction. Hence would the integer $m(n-1)!$ equal an integer plus a fraction. Since this is impossible, e cannot equal a fraction.

EXERCISES.

1. Find the general term of the expansion of $\log_e (1 + x + x^2 + x^3)$.
2. Find the general term of the expansion of $\log_e (1 - 3x + 2x^2)$.
3. If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots$, then $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots$
4. $\log_e (n + c) - \log_e (n - c) = 2\left(\frac{c}{n} + \frac{c^3}{3n^3} + \frac{c^5}{5n^5} + \frac{c^7}{7n^7} + \ldots\right)$.
5. $\log_e 2 = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \ldots$
6. $\log_e 2 = \frac{1}{2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{3 \cdot 4 \cdot 5} + \frac{1}{5 \cdot 6 \cdot 7} + \ldots$
7. $\log_e \frac{r+s}{r-s} = \frac{2rs}{r^2+s^2} + \frac{1}{3}\left(\frac{2rs}{r^2+s^2}\right)^3 + \frac{1}{5}\left(\frac{2rs}{r^2+s^2}\right)^5 + \ldots$
8. $\log_e x = \frac{x-1}{x+1} + \frac{1}{2}\,\frac{x^2-1}{(x+1)^2} + \frac{1}{3}\,\frac{x^3-1}{(x+1)^3} + \ldots$

CHAPTER XVII.

SUMMATION OF SERIES.

137. The method of undetermined coefficients (Chapter VIII) and the method of mathematical induction (Chapter XI) have been employed to effect the summation of certain classes of series. Likewise, the sum of an arithmetical or a geometrical progression has been obtained (Chapter VI). We here discuss methods of greater generality for finding the sum of a series

$$u_1 + u_2 + u_3 + \dots + u_n + \dots$$

In case the series is an infinite series, the result is valid only when the series is convergent.

For an arithmetical progression of common difference d, we have $u_{n-1} - u_{n-2} = d = u_n - u_{n-1}$, whence $u_n - 2u_{n-1} + u_{n-2} = 0$. The latter relation is called the **generating relation**, since the series may be constructed by means of it. Thus, taking $n = 3$ and 4, we get $u_3 = 2u_2 - u_1$, $u_4 = 2u_3 - u_2 = 3u_2 - 2u_1$. The series is therefore $u_1, u_2, 2u_2 - u_1, 3u_2 - 2u_1, \dots$, of common difference $u_2 - u_1$.

For a geometrical progression, the generating relation is $u_n - ru_{n-1} = 0$, r being the common ratio. Inversely, every two-term generating relation leads to a geometrical progression.

138. In general, a series $u_1 + u_2 + u_3 + \dots$ is called a **recurring series of order k,** if it has a **generating relation of order k,**

$$u_n + c_1u_{n-1} + c_2u_{n-2} + \dots + c_ku_{n-k} = 0 \quad (c_1, \dots, c_k \text{ constants}),$$

which holds true for $n = k + 1,\ k + 2, \ldots$, on to the end of the series when the latter is finite, but holds *ad infinitum* when the series is infinite.

In addition to the above examples, consider the series

$$1+2x+3x^2+4x^3+\ldots+(n-2)x^{n-3}+(n-1)x^{n-2}+nx^{n-1}+\ldots$$

Since it is not a geometrical progression, the generating relation, if one exists, must be of order at least two. Trying the value $k = 2$, we have $u_n + c_1u_{n-1} + c_2u_{n-2} = 0$ for $n = 3, 4, \ldots$ Hence

$$nx^{n-1} + c_1(n-1)x^{n-2} + c_2(n-2)x^{n-3} = 0.$$

$$\therefore\ n(x^2 + c_1x + c_2) - c_1x - 2c_2 = 0 \quad (\text{for } n = 3, 4, 5, \ldots).$$

$$\therefore\ x^2 + c_1x + c_2 = 0, \qquad -c_1x - 2c_2 = 0 \qquad (\S\ 78).$$

$$\therefore\ c_1 = -2x, \qquad c_2 = x^2.$$

Hence there is a generating relation $u_n - 2xu_{n-1} + x^2u_{n-2} = 0$ of order two. But this generating relation determines the above special series only when it is also given that $u_1 = 1,\ u_2 = 2x$. In fact, the relation only expresses $u_3, u_4, \ldots$ in terms of u_1 and u_2.

139. In general, a generating relation of order k determines a particular series only when the first k terms of the series are given. On the other hand, if it is known that a series has a generating relation of order k, but the constants $c_1, c_2, \ldots, c_k$ are not known, we must have, in addition to the first k terms of the series, the next k terms in order to determine $c_1, \ldots, c_k$.

If the *general* term is given, the preceding conditions are evidently satisfied and the generating relation is found as in the above example.

EXAMPLE. Find the generating relation of the recurring series

$$1 + x + x^2 + 2x^3 + 3x^4 + 2x^5 + 2x^6 + 7x^7 + \ .$$

If the generating relation were of order two, $u_n + c_1u_{n-1} + c_2u_{n-2} = 0$, we would have $x^2 + c_1x + c_2 = 0$, $2x^3 + c_1x^2 + c_2x = 0$, whence $x = 0$. Assuming, therefore, that the generating relation is of order three,

$$u_n + c_1u_{n-1} + c_2u_{n-2} + c_3u_{n-3} = 0,$$

we have, for $n = 4, 5, 6$, the conditions

$$c_1x^2 + c_2x + c_3 = -2x^3,$$
$$2c_1x^3 + c_2x^2 + c_3x = -3x^4,$$
$$3c_1x^4 + 2c_2x^3 + c_3x^2 = -2x^5.$$

Hence $c_1 = -x$, $c_2 = 2x^2$ $c_3 = -3x^3$. The conditions for $n = 7$ and 8 are seen to be satisfied. Hence $u_n - xu_{n-1} + 2x^2u_{n-2} - 3x^3u_{n-3} = 0$.

140. Problem. *To find the sum of n terms of a recurring series of order two,*

$$S_n \equiv a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{n-1}x^{n-1}.$$

Let the generating relation be $u_n + pxu_{n-1} + qx^2u_{n-2} = 0$. Then

$$pxS_n = pa_0x + pa_1x^2 + \ldots + pa_{n-2}x^{n-1} + pa_{n-1}x^n,$$

$$qx^2S_n = qa_0x^2 + \ldots + qa_{n-3}x^{n-1} + qa_{n-2}x^n + qa_{n-1}x^{n+1}.$$

Since

$$a_2x^2+pa_1x^2+qa_0x^2=0, \ldots, a_{n-1}x^{n-1}+pa_{n-2}x^{n-1}+qa_{n-3}x^{n-1}=0,$$

$$S_n+pxS_n+qx^2S_n=a_0+a_1x+pa_0x+pa_{n-1}x^n+qa_{n-2}x^n+qa_{n-1}x^{n+1},$$

$$\therefore\ S_n = \frac{a_0 + (a_1+pa_0)x + (pa_{n-1}+qa_{n-2})x^n + qa_{n-1}x^{n+1}}{1+px+qx^2}$$

Let x be a value for which the series is convergent, so that (end of § 107)

$$\underset{n \doteq \infty}{\text{limit}}\ a_{n-1}x^{n-1} = 0.$$

Then $a_{n-2}x^n$ and $a_{n-1}x^{n+1}$ will also approach the limit zero. Hence

$$S_\infty = \frac{a_0 + (a_1 + pa_0)x}{1 + px + qx^2}.$$

For a recurring series of order *one* or *three* we find, in a similar manner, that the sum to infinity is, respectively,

$$\frac{a_0}{1+px}, \quad \frac{a_0 + (a_1 + pa_0)x + (a_2 + pa_1 + qa_0)x^2}{1 + px + qx^2 + rx^3},$$

the generating relation being, respectively,

$$u_n + pxu_{n-1} = 0, \qquad u_n + pxu_{n-1} + qx^2u_{n-2} + rx^3u_{n-3} = 0.$$

For the series of § 138, we have $a_0 = 1$, $a_1 = 2$, $p = -2$, $q = 1$.

$$\therefore\ S_\infty = \frac{1}{1 - 2x + x^2} = \frac{1}{(1 - x)^2} = (1 - x)^{-2}.$$

For the series of § 139, $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, $p = -1$, $q = 2$, $r = -3$.

$$\therefore\ S_\infty = \frac{1 + 2x^2}{1 - x + 2x^2 - 3x^3}.$$

141. The fraction obtained as the sum to infinity of a recurring series is called the **generating fraction**, since the series may be obtained by the expansion of the fraction (§ 122).

When the generating fraction can be expressed in terms of partial fractions, whose denominators are powers of binomial expressions, the general term of the recurring series may be readily obtained by the binomial theorem.

For example, the series $1 + 9x - 15x^2 + 57x^3 - 159x^4 + \ldots$ has the generating relation $u_n + 2xu_{n-1} - 3x^2u_{n-2} = 0$, so that (§§ 140, 80)

$$S_\infty = \frac{1 + 11x}{1 + 2x - 3x^2} = \frac{3}{1 - x} - \frac{2}{1 + 3x}$$

$$= 3(1 + x + x^2 + \ldots + x^r + \ldots) - 2(1 - 3x + 9x^2 - \ldots + (-3x)^r + \ldots),$$

in which the coefficient of x^r is $3 - 2(-3)^r$.

EXERCISES.

Find the generating relation, the generating fraction, and the coefficient of x^r, of the recurring series:

1. $1 + 4x - 2x^2 + 10x^3 - 14x^4 + \ldots$
2. $1 - 5x + 34x^2 - 260x^3 + 2056x^4 - \ldots$
3. $5 - 8x + 56x^2 - 176x^3 + 800x^4 + \ldots$
4. $1 + 6x - 4x^2 - 40x^3 - 112x^4 - 32x^5 + 704x^6 + \ldots$
5. $1 - 5x + 9x^2 - 13x^3 + \ldots$
6. $2 + 3x + 5x^2 + 9x^3 + \ldots$
7. $4 - 5x + 7x^2 - 11x^3 + \ldots$
8. $1 + 7x - x^2 + 43x^3 + \ldots$
9. $1^2 + 2^2 + 3^2 + \ldots + n^2.$
10. $1^3 + 2^3 + \ldots + n^3.$

THE METHOD OF DIFFERENCES.

142. If the first term of any series is subtracted from the second term, the second from the third, etc., a series is obtained which is called the **series of the first order of differences.** Proceeding similarly with the latter series we obtain the **series of the second order of differences** of the original series. Similarly, we obtain the series of any order of differences.

Series: $u_1, \quad u_2, \quad u_3, \quad u_4, \dots$
1st order differences: $u_2 - u_1, \quad u_3 - u_2, \quad u_4 - u_3, \dots$
2nd order differences: $u_3 - 2u_2 + u_1, \quad u_4 - 2u_3 + u_2, \dots$
3rd order differences: $u_4 - 3u_3 + 3u_2 - u_1, \dots$

Denote by $D_1, D_2, D_3, \dots$, the first terms of the series of the 1st, 2nd, 3rd, ..., orders of differences, respectively:

$$D_1 = u_2 - u_1, \quad D_2 = u_3 - 2u_2 + u_1, \quad D_3 = u_4 - 3u_3 + 3u_2 - u_1, \dots$$
$$\therefore \quad u_2 = u_1 + D_1, \quad u_3 = u_1 + 2D_1 + D_2, \quad u_4 = u_1 + 3D_1 + 3D_2 + D_3, \dots$$

We observe that the expressions for u_2, u_3, u_4 have the same coefficients as the binomial coefficients in $(a+b)^1$, $(a+b)^2$, $(a+b)^3$. We proceed to prove by mathematical induction that

$$(1) \quad u_{n+1} = u_1 + nD_1 + \frac{n(n-1)}{1 \cdot 2} D_2 + \dots + {}_nC_r D_r + \dots + D_n.$$

Suppose that the theorem has been verified for any series as far as the $(m+1)$st term (as has been done for $m = 1, 2, 3$). Hence

$$(2) \quad u_{m+1} = u_1 + mD_1 + {}_mC_2 D_2 + {}_mC_3 D_3 + \dots + {}_mC_r D_r + \dots + D_m.$$

Writing the analogous formula for the series of the first order of differences $u_2 - u_1 = D_1$, $u_3 - u_2, \dots, u_{m+1} - u_m$, $u_{m+2} - u_{m+1}, \dots$, the first terms of whose 1st, 2nd, 3rd, ..., orders of differences are $D_2, D_3, D_4, \dots$, respectively, we get

$$(3) \quad u_{m+2} - u_{m+1}$$
$$= D_1 + mD_2 + {}_mC_2 D_3 + {}_mC_3 D_4 + \dots + {}_mC_{r-1} D_r + \dots + D_{m+1}.$$

Adding (2) and (3), and applying ${}_mC_r + {}_mC_{r-1} = {}_{m+1}C_r$ (§ 89), we get

$$u_{m+1} = u_1 + (m+1)D_1 + {}_{m+1}C_2 D_2 + {}_{m+1}C_3 D_3 + \dots + {}_{m+1}C_r D_r + \dots + D_{m+1}.$$

But this is formula (1) for the case $n = m + 1$. Hence, if the law holds for $n = m$, it holds for $n = m + 1$. But the law holds for $n = 1, 2, 3$. It therefore holds for any positive integral value of n.

EXAMPLE. For the series 5, 8, 11, 14, . . . , the series of the first order of differences is 3, 3, 3, . . . ; that of the second order, 0, 0, . . . Hence $D_1 = 3$, $D_2 = 0$, $D_3 = 0$, . . . , $D_r = 0$. Hence $u_{n+1} = 5 + 3n$; this result may also be derived as the $(n + 1)$st term of an A. P. of common difference 3.

143. Problem: *To find the sum S_n of the first n terms of the series*

$$(4) \qquad v_1, \quad v_2, \quad v_3, \quad v_4, \dots, v_n, \dots$$

Evidently S_n is the $(n + 1)$st term of the series

$$(5) \quad 0, \quad v_1, \quad v_1 + v_2, \quad v_1 + v_2 + v_3, \dots, \quad v_1 + v_2 + \dots + v_n, \dots$$

The series of the first order of differences of (5) is clearly (4). Hence if $d_1, d_2, d_3, \dots$, are the first terms of the series of the 1st, 2nd, 3rd, . . . , orders of differences of (4), then $v_1, d_1, d_2, d_3, \dots$, are the first terms of the series of the 1st, 2nd, 3rd, 4th, . . . , orders of differences of (5). Hence the $(n + 1)$st term of series (5) is [by formula (1), with $u_1 = 0$, $D_1 = v_1$, $D_2 = d_1$, $D_3 = d_2, \dots$]

$$S_n = nv_1 + \frac{n(n-1)}{1\cdot 2}d_1 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d_2 + \dots + {}_nC_r d_{r-1} + \dots + d_{n-1}.$$

144. EXAMPLE 1. Find the nth term and the sum of the first n terms of

	1, 15, 40, 79, 135, 211, . . .
1st order differences:	14, 25, 39, 56, 76, . . .
2nd order differences:	11, 14, 17 20, . . .
3rd order differences:	3, 3, 3, . . .
4th order differences:	0, 0, . . .

Hence the nth term and the sum S_n are respectively

$$1 + (n-1)14 + \frac{(n-1)(n-2)}{1\cdot 2}11 + \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}3 = \tfrac{1}{2}(n^3 + 5n^2 + 6n - 10),$$

$$S_n = n + \frac{n(n-1)}{1\cdot 2}14 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}11 + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}8$$

$$= \tfrac{1}{24}n(3n^3 + 26n^2 + 69n - 74).$$

EXAMPLE 2. Find the sum of the squares of the first n integers.

Series: $1,\ 4,\ 9,\ 16,\ \dots,\ r^2, (r+1)^2,\ (r+2)^2,\ (r+3)^2, \dots$

1st order differences: $3,\ 5,\ 7,\ \dots\dots,\ 2r+1,\ 2r+3,\ 2r+5, \dots$

2nd order differences: $2,\ 2,\ \dots\dots\dots\dots,\ 2,\quad 2, \dots$

3rd order differences: $0,\ \dots\dots\dots\dots\dots,\ 0, \dots$

$$\therefore\quad S_n = n + \frac{n(n-1)}{1\cdot 2}3 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}2 = \frac{n}{6}(n+1)(2n+1).$$

EXAMPLE 3. Find the number of shot arranged in a complete pyramid whose base is an equilateral triangle.

The respective layers contain the following number of shot:

$1,\ 3,\ 6,\ 10,\ \dots,\ (1+2+\dots+r),\ (1+2+\dots+\overline{r+1}),\ (1+\dots+\overline{r+2}),\ (1+\dots+\overline{r+3}), \dots$

1st order: $2,\ 3,\ 4,\ \dots\dots\dots,\ r+1,\quad r+2,\quad r+3, \dots$

2nd order: $1,\ 1,\ \dots\dots\dots\dots,\ 1,\quad 1, \dots$

3rd order: $0,\ \dots\dots\dots\dots\dots,\ 0, \dots$

$$\therefore\quad S_n = n + \frac{n(n-1)}{1\cdot 2}2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} = \tfrac{1}{6}n(n+1)(n+2).$$

EXERCISES.

1. Find the sum of the first n natural numbers.

2. Find the ninth term and the sum of the first nine terms of 2, 9, 28, 65, 126, . . .

Find the nth term and the sum of the first n terms of

3. 1, 8, 25, 52, 89, . . .
4. 2, 12, 28, 50, 78, 112, . . .
5. 1, 11, 35, 79, 149, 251, . . .
6. 2, 20, 48, 86, 134, 192, . . .
7. 1, 4, 10, 20, 35, 56, . . .
8. 4, 8, 0, −32, −100, −216, . . .
9. $1^3, 2^3, 3^3, 4^3, \dots$
10. $1^4, 2^4, 3^4, 4^4, \dots$
11. $1\cdot 3,\ 4\cdot 7,\ 9\cdot 13,\ 16\cdot 21, \dots$
12. $1\cdot 2,\ 2\cdot 4,\ 4\cdot 7,\ 8\cdot 11,\ 16\cdot 16, \dots$

Find the number of shot in

13. A pyramid of triangular base each of whose sides contains 20 shot.

14. A pyramid of square base, 15 shot in a side.

15. A truncated pyramid of 30 layers, with 50 shot in a side of the square base.

16. A truncated pyramid, whose base is an equilateral triangle with 100 shot in a side, and whose top has 56 shot in a side.

17. A complete rectangular pile with base 9 by 15 shot.

18. A complete rectangular pile with 15 layers and 20 shot in the longer side of the base.

19. A complete rectangular pile of n layers with m shot in the top layer contains $\frac{1}{6}n(n+1)(3m+2n-2)$ shot.

CHAPTER XVIII.

GRAPHIC ALGEBRA; SIMULTANEOUS EQUATIONS; SIMULTANEOUS INEQUALITIES; SOLUTION OF NUMERICAL EQUATIONS.

145. The numerical results of an investigation are often exhibited in a table of statistics accompanied by a diagram. For a comparison of the results, the diagram shows at a glance what would be gotten from the table only after a minute study.

Suppose that the number of inches of rainfall in the 1st, . . . , 12th months of a given year was 2, 1, 0.5, 3, 4, 3.5, 1, 0, 2, 1.5, 0.5, 1 respectively. We may represent these figures graphically by marking off the proper number of vertical divisions above the base line at the point indicating the corresponding month (Fig. 1).

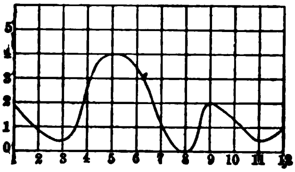

FIG. 1.

Suppose that a quantity y varies as the quantity x in such a way that $y = 2x$. Then for $x = 0$, $y = 1$; for $x = 1$, $y = 2$; for $x = 2$, $y = 4$; for $x = -1$, $y = -2$; for $x = -2$, $y = -4$; etc.

We may represent this relation graphically as in Fig. 2, in which the values of x are measured along the horizontal line OX (the positive values to the right of O and the negative values to the left of O) and the corresponding values of y are measured in the vertical direction (upwards or downwards, according as the value is positive or negative). It is easily shown that the points so located all lie on a straight line AOB [see § 151].

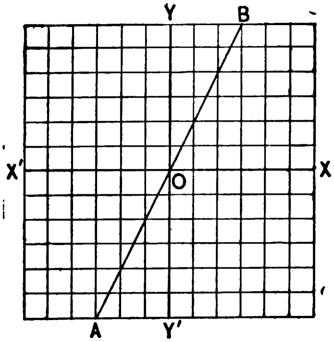

FIG. 2.

146. Definitions.* The two perpendicular lines $X'OX$ and $Y'OY$ are called the **x-axis** and the **y-axis** respectively; together they are the **axes of coördinates**; their intersection O is called the **origin of coördinates.** To **plot** the point whose coördinates are x and y, we mark off x units on the x-axis from the origin O, and on the vertical line (parallel to the y-axis) through the point thus reached we mark off y units; the final point reached is designated as the point (x, y). In Fig. 3, we have plotted four points whose pairs of coördinates differ only in their signs.

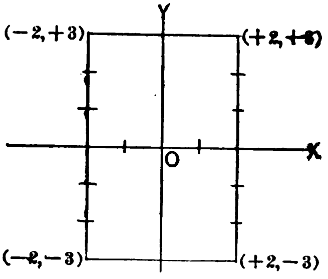

FIG. 3.

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* The system was introduced by Descartes in 1637.

Since there are innumerable pairs of values x, y which satisfy the equation $y = 2x$, there will be a boundless number of corresponding points (x, y) which lie on a line, called the **graph** of the equation $y = 2x$. It would appear from Fig. 2 that the graph of $y = 2x$ is a straight line AOB [compare § 151].

147. The graph of the equation $x^2 + y^2 = 25$ will be a circle of radius 5 and center at the origin O. Indeed, if we join O with the plot P of any point (x, y) such that $x^2 + y^2 = 25$, OP is the hypothenuse of a right-angled triangle whose base is x and vertical side is y, so that $(OP)^2 = x^2 + y^2 = 25$, $OP = 5$. Hence every such point P lies at the distance 5 from O. The circle is given in Fig. 4.

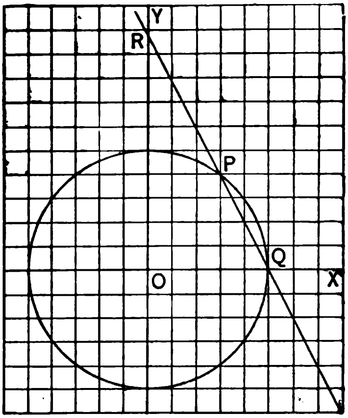

FIG. 4.

The graph of the equation $2x + y = 10$ is a straight line (§ 151). Hence two points will determine it; for example, (0, 10) and (5, 0), whose plots are R and Q in Fig. 4.

It appears from the figure that the circle and the straight line RQ intersect in the points P and Q, the plots of (3, 4) and (5, 0).

The corresponding algebraic problem consists in determining all sets of values x, y which satisfy both of the equations

$$x^2 + y^2 = 25, \quad 2x + y = 10.$$

We must therefore consider the two equations to be simultaneous (§ 34). Setting $y = 10 - 2x$ in the first equation, we get

$$x^2 - 8x + 15 = 0, \quad x = 3 \text{ or } 5.$$

Hence the only sets of solutions are $x = 3$, $y = 4$; $x = 5$, $y = 0$.

148. In general, the problem to solve two simultaneous equations in x, y is equivalent to the problem to find the coördinates of the points of intersection of the graphs of the two equations.

In particular, two simultaneous equations of the first degree in x and y have at most one set of solutions in common. This result follows algebraically from the solution by determinants (§ 35), and geometrically from the fact that the two graphs are straight lines (§ 151). For example, the simultaneous equations

$$2x + y = 10, \quad 4x - 3y = 0,$$

have only the common set of solutions $x = 3$, $y = 4$; their graphs, the straight lines BQ and OP (Fig. 4), have only the common point P, viz., (3, 4).

149. If we attempt to solve by determinants the equations

$$2x + y = 10, \quad 2x + y = 4,$$

we get $0 \cdot x = 6$, $0 \cdot y = -12$, so that there are no common sets of finite solutions. Plotting the graph of $2x + y = 4$, we obtain a straight line which apparently does not intersect the graph of $2x + y = 10$. The fact that the graphs are parallel lines may be proved by observing that the difference between the y-coördinates corresponding to the same x-coördinate is always 6. Hence the graphs have no point of intersection in the finite part of the plane.

EXERCISES.

1. Prove that the graphs of $x^2 + y^2 = 25$ and $2x + y = \sqrt{125}$ are tangent

2. Do the graphs of $x^2 + y^2 = 25$ and $2x + y = 15$ intersect ?

3. Find the intersections of the graphs of $x^2 + y^2 = 25$ and $x - 5y = 0$.

4. Find the point common to the graphs of the three equations

$$2x - 3y = 7, \quad 3x - 4y = 13, \quad 8x - 11y = 33.$$

5. Find the intersection of the graphs of $x^2 + y^2 = 9$ and $x + y = 2$.

6. Show that a parallelogram is formed by the graphs of

$$\frac{x}{a}+\frac{y}{b}=1,\quad \frac{x}{b}+\frac{y}{a}=1,\quad \frac{x}{a}+\frac{y}{b}=2,\quad \frac{x}{b}+\frac{y}{a}=2.$$

150. Theorem. *The area of the triangle whose vertices are the points* (x_1, y_1), (x_2, y_2), *and* (x_3, y_3) *equals the determinant*

$$\begin{vmatrix} x_1 & y_1 & \frac{1}{2} \\ x_2 & y_2 & \frac{1}{2} \\ x_3 & y_3 & \frac{1}{2} \end{vmatrix}.$$

Let the three vertices be P_1, P_2, P_3, respectively. Evidently, triangle $P_1P_2P_3$ equals
trapezium $P_1Q_1Q_3P_3$ – trapezium $P_1Q_1Q_2P_2$ – trapezium $P_2Q_2Q_3P_3$.

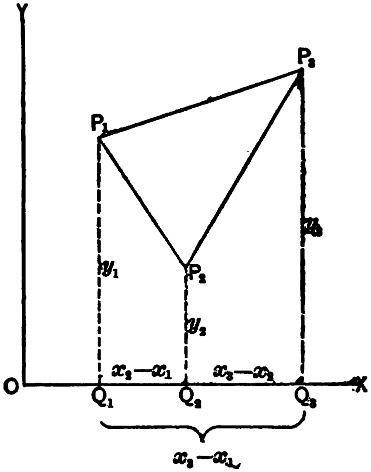

Fig. 5.

But, by geometry, the area of a trapezium equals one half the sum of the two parallel sides multiplied by the perpendicular distance between them. Since $x_1 = OQ_1$, $x_2 = OQ_2$, we have $x_2 - x_1 = Q_1Q_2$; similarly, $x_3 - x_2 = Q_2Q_3$, $x_3 - x_1 = Q_1Q_3$. Hence

$$\begin{aligned}\triangle &= \tfrac{1}{2}(y_3+y_1)(x_3-x_1) - \tfrac{1}{2}(y_2+y_1)(x_2-x_1) - \tfrac{1}{2}(y_3+y_2)(x_3-x_2)\\ &= \tfrac{1}{2}\{x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + x_3y_1 - x_1y_3\}.\end{aligned}$$

Using the determinant notation (§ 36), we get

$$(1)\qquad \Delta = \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

For example, the area of the triangle with the vertices (0, 2), (3, 0), (4, 1) is

$$\tfrac{1}{2}\begin{vmatrix} 0 & 2 & 1 \\ 3 & 0 & 1 \\ 4 & 1 & 1 \end{vmatrix} = \tfrac{5}{2}.$$

151. Theorem. *The graph of an equation of the first degree*

$$(2)\qquad Ax + By + C = 0$$

is always a straight line.

Consider three points (x_1, y_1), (x_2, y_2), (x_3, y_3) of the graph. Then

$$(3)\quad Ax_1 + By_1 + C = 0,\quad Ax_2 + By_2 + C = 0,\quad Ax_3 + By_3 + C = 0.$$

The area of the triangle whose vertices are the three points is given by formula (1). If $A \neq 0$, we multiply the elements of the first column of the determinant by A, the elements of the second column by B, the elements of the third column by C, and add the last two sets of products to the first set. We have (§§ 45, 47)

$$A\,\Delta = \tfrac{1}{2}\begin{vmatrix} Ax_1 + By_1 + C & y_1 & 1 \\ Ax_2 + By_2 + C & y_2 & 1 \\ Ax_3 + By_3 + C & y_3 & 1 \end{vmatrix} = \tfrac{1}{2}\begin{vmatrix} 0 & y_1 & 1 \\ 0 & y_2 & 1 \\ 0 & y_3 & 1 \end{vmatrix} = 0,$$

upon applying the hypotheses (3). Hence $\Delta = 0$, so that the three points lie in a straight line. Let the third point (x_3, y_3) move to any new position on the graph. As the area is again zero, the new point lies on the straight line determined by (x_1, y_1) and (x_2, y_2). Hence *every* point of the graph lies on that straight line.

If $A = 0$ and $B \neq 0$, we multiply the elements of the second column by B and add to them the products of the elements of the third column by C. The resulting determinant has three zeros in the second column, so that $B\,\Delta = 0$, whence $\Delta = 0$. In this case the line (2) becomes $By + C = 0$ and is parallel to the x-axis, since the y-coördinate remains constant.

It remains to prove that, inversely, every point on the straight line joining (x_1, y_1) and (x_2, y_2) lies on the graph of (2). This result is shown in § 153 below.

152. *To find the coördinates of the point which divides in a given ratio the straight line joining two given points.*

Let the first point P_1 be (x_1, y_1) and the second point P_2 be (x_2, y_2). Let the given ratio be $m_1 : m_2$. Let P_i be the point which divides P_1P_2 *internally* in this ratio, and P_e be the point which divides P_1P_2 *externally* in this ratio. Let their coördinates be (x_i, y_i) and (x_e, y_e), respectively.

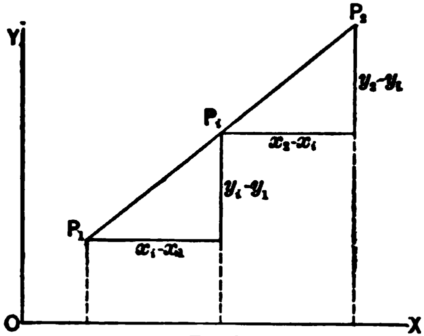

FIG. 6.

Since $P_1P_i : P_2P_i = m_1 : m_2$ and $P_1P_e : P_2P_e = m_1 : m_2$, we have

$$\frac{y_i - y_1}{y_2 - y_i} = \frac{x_i - x_1}{x_2 - x_i} = \frac{m_1}{m_2}, \qquad \frac{y_1 - y_e}{y_2 - y_e} = \frac{x_1 - x_e}{x_2 - x_e} = \frac{m_1}{m_2}.$$

Hence

$$(4) \qquad x_i = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}, \qquad y_i = \frac{m_1y_2 + m_2y_1}{m_1 + m_2};$$

$$(5) \qquad x_e = \frac{m_1x_2 - m_2x_1}{m_1 - m_2}, \qquad y_e = \frac{m_1y_2 - m_2y_1}{m_1 - m_2}.$$

Corollary. *The middle point of the straight line joining* (x_1, y_1) *and* (x_2, y_2) *is*

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right). \tag{6}$$

Note. *The distance between two points* (x_1, y_1) *and* (x_2, y_2) *is*

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}. \tag{7}$$

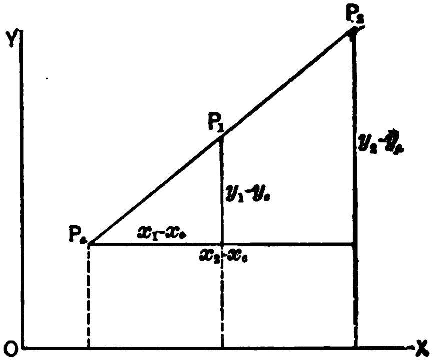

FIG. 7.

153. We may now prove that if the graph of $Ax+By+C=0$ contains two distinct points (x_1, y_1) and (x_2, y_2), it contains every point on the straight line joining them. By hypothesis, the first and second relations (3) are satisfied. In view of the last section, it remains to be proved that the points P_i and P_e with the coördinates (4) and (5), respectively, lie on the graph of $Ax+By+C=0$, whatever be the values of m_1 and m_2. The condition for P_i is that

$$Ax_i+By_i+C \equiv \frac{m_1}{m_1+m_2}(Ax_2+By_2)+\frac{m_2}{m_1+m_2}(Ax_1+By_1)+C$$

shall vanish. In view of the first and second relations (3), it equals

$$\frac{m_1}{m_1+m_2}(-C)+\frac{m_2}{m_1+m_2}(-C)+C \equiv 0.$$

Similarly, Ax_e+By_e+C vanishes.

154. From what precedes, every straight line is the graph of some equation of the first degree. If two points (x_1, y_1) and (x_2, y_2) on the line are given, the required equation may be written

$$(8) \qquad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

Indeed, upon expansion according to the elements of the first row (§ 46), there results an equation of the first degree in x and y, which must therefore represent some straight line (§ 151). But the line coincides with the given straight line. In fact, it contains the point (x_1, y_1), since the replacement of x by x_1 and y by y_1 in the determinant in equation (8) gives a determinant having the first and second rows alike, so that the equation is satisfied (§ 46). Similarly, it contains the point (x_2, y_2).

A second proof follows from § 150, since the area of the triangle formed by (x, y), (x_1, y_1), (x_2, y_2), is one-half the determinant in (8); but the area must be zero if the point (x, y) lies on the line joining (x_1, y_1) and (x_2, y_2).

155. EXAMPLE 1. Find the equation of the straight line making an intercept a on the x-axis and an intercept b on the y-axis.

The line through the points $(a, 0)$ and $(0, b)$ has the equation

$$\begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} \equiv ab - ay - bx = 0.$$

It may be written in the convenient form $\frac{x}{a} + \frac{y}{b} = 1$.

EXAMPLE 2. The medians of any triangle meet in a point and each median is trisected by that point.

Let the vertices P_1, P_2, P_3 be (x_1, y_1), (x_2, y_2), (x_3, y_3), respectively. By the corollary of § 152, the middle point M_{12} of the side P_1P_2 is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. Let Q be the point which divides the sect P_3M_{12} internally in the ratio 2 : 1. By formula (4), the coördinates of Q are

$$x_4 \equiv \frac{2\left(\frac{x_1 + x_2}{2}\right) + 1(x_3)}{2 + 1} = \frac{x_1 + x_2 + x_3}{3},$$

$$y_t \equiv \frac{2\left(\frac{y_1+y_2}{2}\right)+1(y_3)}{2+1} = \frac{y_1+y_2+y_3}{3}.$$

Similarly, the middle point M_{13} of the side P_1P_3 is $\left(\frac{x_1+x_3}{2},\ \frac{y_1+y_3}{2}\right)$, and the coördinates of the point which divides the sect P_2M_{13} internally in the ratio 2 : 1 are seen to be the values x_t, y_t just given. In a similar manner, or by the symmetry of the result, the point (x_t, y_t) divides the sect P_1M_{23} internally in the ratio 2 : 1, M_{23} being the middle point of the side P_2P_3. Hence the medians meet in a point and trisect each other.

EXAMPLE 3. Find the equations to the straight lines which pass through the point (1, – 3) and make equal intercepts on the two axes.

By Ex. 1, the required lines have the equations

$$\frac{x}{a}+\frac{y}{-a}=1, \quad \frac{x}{-b}+\frac{y}{-b}=1,$$

since a line through (1, – 3) will either have a positive intercept on the x-axis and a negative intercept on the y-axis or else have negative intercepts on both axes (as a diagram will show). Substituting 1 for x and – 3 for y, we have for the respective lines the conditions

$$\frac{1}{a}+\frac{-3}{-a}=1, \quad \frac{1}{-b}+\frac{-3}{-b}=1,$$

whence $a = 4$, $b = 2$. The lines are $x - y = 4$ and $x + y = -2$.

EXAMPLE 4. To find the coördinates of the center of the circle inscribed in a given triangle.

Let the vertices P_1, P_2, P_3 be (x_1, y_1), (x_2, y_2), (x_3, y_3) and the opposite sides s_1, s_2, s_3. The bisector of the interior angle at P_1 meets the side P_2P_3 at a point Q_1 which divides it internally in the ratio $s_3 : s_2$. Hence the coördinates of Q_1 are (§ 152)

$$\frac{s_3x_3+s_2x_2}{s_3+s_2}, \quad \frac{s_3y_3+s_2y_2}{s_3+s_2}.$$

But $P_2Q_1 : P_3Q_1 = s_3 : s_2$ gives $P_2Q_1 + P_3Q_1 : P_3Q_1 = s_3 + s_2 : s_2$, whence

$$P_3Q_1 = \frac{s_1s_2}{s_3+s_2} \equiv \sigma.$$

The bisector of the interior angle at P_3 meets P_1Q_1 at the required center C. Hence C divides P_1Q_1 internally in the ratio $s_2 : P_3Q_1$, viz., $s_2 : \sigma$. Hence the coördinates of C are

$$x_c = \frac{s_2\left(\frac{s_3x_3+s_2x_2}{s_3+s_2}\right)+\sigma x_1}{s_2+\sigma}, \quad y_c = \frac{s_2\left(\frac{s_3y_3+s_2y_2}{s_3+s_2}\right)+\sigma y_1}{s_2+\sigma}.$$

$$\therefore \quad x_c = \frac{s_1x_1+s_2x_2+s_3x_3}{s_1+s_2+s_3}, \quad y_c = \frac{s_1y_1+s_2y_2+s_3y_3}{s_1+s_2+s_3}.$$

Thus, if the vertices are (0, 12), (5, 0), (35, 0), we get (Note, § 152) $s_1 = 30$, $s_2 = 37$, $s_3 = 13$. The coördinates of C are therefore $x_c = 8$, $y_c = 4\frac{1}{2}$.

EXERCISES.

Find the equation of the straight line

1. Cutting off 3 on the x-axis and 4 on the y-axis.

2. Cutting off -2 on the x-axis and -5 on the y-axis.

3. Passing through the points (0, 0) and (3, -3).

4. Passing through the points (5, 6) and (3, 4).

5. Passing through the point (1, 4) and cutting off -3 on the y-axis.

6. Find the equations of the diagonals of the rectangle formed by the lines $x = 2$, $x = -3$, $y = 4$, $y = -5$.

7. Find the equations of the lines through the origin and the points trisecting the segment of the line $3x + y - 12$ intercepted between the axes.

8. Show that the coördinates of the centers C_1, C_2, C_3 of the circles escribed to the triangle with the vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) and the opposite sides s_1, s_2, s_3, are

$$\frac{-s_1x_1 + s_2x_2 + s_3x_3}{-s_1 + s_2 + s_3}, \quad \frac{-s_1y_1 + s_2y_2 + s_3y_3}{-s_1 + s_2 + s_3} \quad \text{(for } C_1 \text{ opposite } s_1\text{)},$$

$$\frac{s_1x_1 - s_2x_2 + s_3x_3}{s_1 - s_2 + s_3}, \quad \frac{s_1y_1 - s_2y_2 + s_3y_3}{s_1 - s_2 + s_3} \quad \text{(for } C_2 \text{ opposite } s_2\text{)},$$

with similar expressions for C_3, the sign of s_3 being changed.

9. Find the centers of the inscribed and escribed circles for the triangle whose vertices are (9, 0), (0, 12) and (0, 40).

10. Find the centers of the inscribed and escribed circles for the triangle whose sides have the equations $4x - 3y = 0$, $3x - 4y + 12 = 0$, $3x + 4y + 2 = 0$.

11. Find the equations of the sides of the triangle with the vertices $(-1, -2)$, $(2, -3)$, $(1, 4)$.

Find the coördinates of the points which divide internally and externally

12. the line joining (1, 2) and (4, 5) in the ratio 2 : 3.

13. the line joining $(-2, -3)$ and $(-4, 7)$ in the ratio 5 : 6.

14. the line joining (0, 0) and (6, 8) in the ratio 4 : 5.

15. the line joining $(c + d,\ c - d)$ and $(c - d,\ c + d)$ in the ratio $c : d$.

16. The lines joining the middle points of opposite sides of a quadrilateral and the line joining the middle points of its diagonals meet in a point and bisect one another.

17. If $P_1, P_2, \ldots, P_n$ denote the points (x_1, y_1), (x_2, y_2), . . . , (x_n, y_n), and P_1P_2 is bisected at the point M_1; M_1P_3 is divided at M_2 in the ratio 1 : 2; M_2P_4 is divided at M_3 in the ratio 1 : 3; M_3P_5 is divided at M_4 in the ratio 1 : 4; etc.; prove that the final point M_{n-1} has the coördinates

$$\frac{1}{n}(x_1 + x_2 + x_3 + \ldots + x_n), \quad \frac{1}{n}(y_1 + y_2 + y_3 + \ldots + y_n).$$

18. Show that the area of the polygon $P_1P_2 \ldots P_n$ of Ex. 17 is

$$\tfrac{1}{2}\left\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+ \qquad +(x_{n-1}y_n+x_ny_{n-1})+(x_ny_1-x_1y_n)\right\}.$$

19. Find the area of the polygon whose vertices are $(4, -7)$, $(5, -2)$, $(3, 4)$, $(1, 1)$.

20. Find the area of the triangle with the vertices (a, b), $(-a, b+c)$, $(-a, c-b)$.

21. Find the coördinates of the point of intersection of the two lines joining the middle points of the opposite sides of the quadrilateral whose vertices, taken in order, are $(0, 3)$, $(0, 0)$, $(4, 0)$, and $(6, 7)$.

156. To find the graph of the equation $y^2 = 4x$, we observe that a negative value of x gives an imaginary value of y, so that the graph lies to the right of the y-axis. For each positive value of x there are two values of y, namely $2\sqrt{x}$ and $-2\sqrt{x}$, so that the graph is symmetrical with respect to the x-axis. Plotting the pairs of points $(1, +2)$, $(1, -2)$, $(2, 2\sqrt{2})$, $(2, -2\sqrt{2})$, $(4, 4)$, $(4, -4)$, $(16, 8)$, $(16, -8)$, . . . , and the point $(0, 0)$, we may trace the graph AOB in Fig. 8. The graph is called a **parabola.**

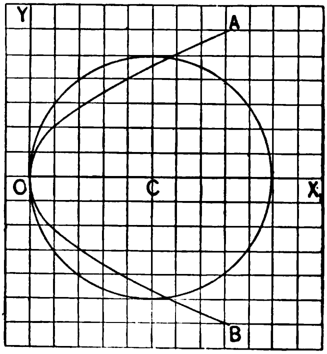

FIG. 8.

The figure also gives the graph of the equation $x^2 + y^2 = 10x$, which may be written $(x-5)^2 + y^2 = 25$. Since the latter is derived from $X^2 + y^2 = 25$, the circle of Fig. 4, by writing $x - 5$ in place of X, it is likewise a circle of radius 5, but referred to a set of axes of coördinates obtained by moving the previous y-axis 5 units to the left. Hence the graph of $x^2 + y^2 = 10x$ is a circle of radius 5 and center (5, 0). Its intersections with the parabola $y^2 = 4x$ are seen to be the three points (0, 0), $(6, \sqrt{24})$, $(6, -\sqrt{24})$, found by solving the equations considered to be simultaneous.

Similarly, $y^2 = 4x$ intersects the circle $x^2 + y^2 = 25$ in just two points; it intersects the circle $(x-4)^2 + y^2 = 25$, with radius 5 and center (4, 0), in four points; it intersects the circles $(x-c)^2 + y^2 = 25$, of radius 5 and center $(c, 0)$, in no point if $c > 7\frac{1}{4}$. In fact, if we eliminate y^2 from the latter equation by means of $y^2 = 4x$, we get a quadratic for x, with the roots

$$c + 2 \pm \sqrt{29 - 4c}.$$

In the limiting case $c = 7\frac{1}{4}$, there is a single point of intersection.

Two simultaneous equations of the second degree in x and y may have 4, 3, 2, 1, *or no sets of real solutions x, y.*

157. In order that the three simultaneous equations

$$(9) \quad a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0, \quad a_3x + b_3y + c_3 = 0$$

shall have a common set of solutions x, y, the determinant

$$D \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

must vanish. This follows from § 44 when we set $z = 1$. To give another proof, multiply the elements of the first column by x and the elements of the second column by y and add the results to the corresponding elements of the third column; the new elements of the third column will then be zero, if x and y be the common solutions. Hence must $D = 0$.

Inversely, if $D = 0$, equations (9) have at least one set of solutions x, y in common (§ 44). In fact, the third equation (9) then follows from the first and second equations. Their graphs are straight lines which, in general, intersect in a point (x, y). But the second and third equations both follow from the first if all the minors of D vanish (§ 44), in which case there are innumerable sets of solutions x, y in common. The three graphs are then coincident lines.

For example, the lines (Fig. 9)

$$\begin{aligned} RP&: \quad 2x + y - 10 = 0,\\ OP&: \quad 4x - 3y = 0,\\ SP&: \quad 2x - y - 2 = 0 \end{aligned}$$

intersect in the point P, viz., (3, 4). The determinant D is seen to be zero.

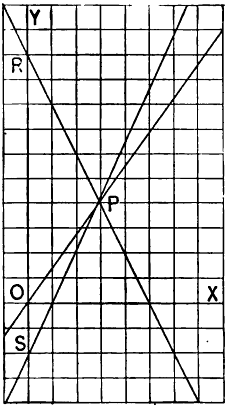

FIG. 9.

158. Theorem. *Two points* $(x_1 y_1)$ *and* $(x_2 y_2)$ *are on the same side or on opposite sides of the graph of* $Ax + By + C = 0$ *according as the quantities*

$$Q_1 \equiv Ax_1 + By_1 + C, \qquad Q_2 \equiv Ax_2 + By_2 + C$$

are of the same sign or of opposite signs.

Through (x_1, y_1) draw a parallel to the y-axis and let it meet the graph of the given line in the point (x_1, y_1') so that,

$$Ax_1 + By_1' + C = 0, \quad y_1' = -(Ax_1 + C) \div B.$$

$$\therefore \quad y_1 - y_1' = (Ax_1 + By_1 + C) \div B \equiv Q_1/B.$$

Through (x_2, y_2) draw a parallel to the y-axis and let it meet the given line in the point (x_2, y_2'). As before, $y_2 - y_2' = Q_2/B$.

Then (x_1, y_1) and (x_2, y_2) are on the same side or on opposite sides of the given line according as $y_1 - y_1'$ and $y_2 - y_2'$ are of the same sign or of opposite signs, i.e., according as Q_1 and Q_2 are of the same sign or of opposite signs.

Corollary. The point (x_1, y_1) and the origin (0, 0) are on the same side of the graph of $Ax + By + C = 0$ if, and only if, $Ax_1 + By_1 + C$ and C have the same sign.

159. A point (x_1, y_1) lies inside, on, or outside a circle of radius r and center at the origin (0, 0) according as

$$x_1^2 + y_1^2 - r^2 < 0, \quad = 0, \quad \text{or} \quad > 0.$$

In fact, the equation of the circle is $x^2 + y^2 = r^2$ (compare § 147).

A point (x_1, y_1) lies inside, on, or outside the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(called an **ellipse**) according as $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 < 0, \quad = 0, \quad \text{or} \quad > 0.$

In proof, draw through the point (x_1, y_1) a parallel to the y-axis and let it meet the graph in a point (x_1, y_1'), so that

$$\frac{x_1^2}{a^2} + \frac{y_1'^2}{b^2} = 1.$$

$$\therefore \quad y_1^2 - y_1'^2 = b^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right).$$

For the parabola $y^2 = 4x$ whose graph AOB is given in Fig. 8, we will say that a point lies within the parabola when it lies on the same side of the graph as the line OX. By a proof similar to that just employed, we find that the point (x_1, y_1) lies within, on, or without the graph of $y^2 = 4x$ according as $y_1^2 - 4x_1$ is less than, equal to, or greater than zero.

160. The preceding results enable us to treat very simply problems in simultaneous inequalities.

EXAMPLE 1. Resolve the simultaneous inequalities

$$f_1 \equiv x - y + 1 > 0, \quad f_2 \equiv 2x + y - 6 > 0, \quad f_3 \equiv -5x + 2y + 10 > 0.$$

The graphs of $f_1 = 0$, $f_2 = 0$, $f_3 = 0$ are given in the accompanying figure. By the corollary of § 158, f_1 is positive for all points on that side of the graph of $f_1 = 0$ on which the origin lies. Similarly for f_3; but f_2 is positive only for the points on the opposite side of $f_2 = 0$ from the origin. The three inequalities are therefore satisfied simultaneously by all the points in the shaded triangle and by no other points. The vertices of this triangle are $(\frac{5}{3}, \frac{8}{3})$, $(4, 5)$, $(\frac{22}{9}, \frac{10}{9})$. Hence the solutions x, y are such that

$$\tfrac{5}{3} < x < 4, \quad \tfrac{10}{9} < y < 5.$$

EXAMPLE 2. Resolve the simultaneous inequalities

$$y^2 - 4x > 0, \quad x^2 - 10x + y^2 < 0.$$

The sets of values x, y must be such that the corresponding points (x, y) lie outside of the parabola $y^2 = 4x$ and inside of the circle $(x - 5)^2 + y^2 = 25$. These points therefore lie within the two crescent-shaped areas in the figure of § 156. Hence the solutions are

$$0 < x < 6, \quad -\sqrt{24} < y < \sqrt{24}, \quad y^2 > 4x.$$

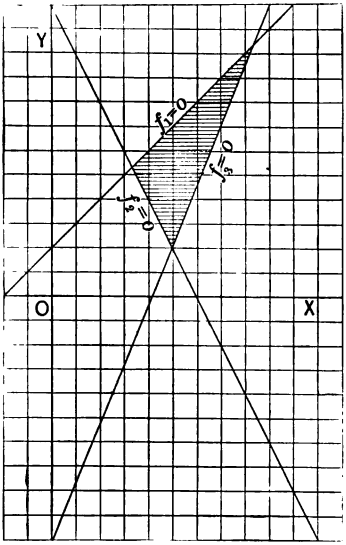

FIG. 10.

EXERCISES.

Plot the graphs and find the common solutions of

1. $x^2 + y^2 = 25, \quad y^2 = 4x + 24.$ **2.** $x^2 + y^2 = 25, \quad y^2 = 4x + 28.$

3. $x^2 + y^2 = 25, \quad \dfrac{x^2}{36} + \dfrac{y^2}{9} = 1.$ **4.** $x^2 + y^2 = 16, \quad x^2 + 4y^2 = 64.$

5. $3x + 3y + 5 = 0, \quad 3x + 4y + 6 = 0, \quad 6x + 5y + 9 = 0.$

6. $\dfrac{x}{a} + \dfrac{y}{b} = 1, \quad \dfrac{x}{b} + \dfrac{y}{a} = 1, \quad y - x = 0.$

7. $x - 2y - 3 = 0, \quad 4y + 6 = 2x.$ **8.** $x^2 + y^2 = 9, \quad 4x^2 + 9y^2 = 36.$

Solve graphically the simultaneous inequalities

9. $x - y > 0, \quad -3x + 2y + 6 > 0, \quad x + 2y - 4 > 0.$

10. $x - y + 4 > 0, \quad x - 2y + 5 < 0, \quad x + 5y - 4 < 0.$

11. $x + y - 3 > 0, \quad 2x - 3y + 1 > 0, \quad 2x - 3y - 5 < 0.$

12. $x^2 + y^2 - 25 < 0, \quad 2x + y - 10 > 0.$

13. $x^2 + y^2 - 25 < 0, \quad 2x + y - 10 < 0, \quad x + y - 2 > 0.$

14. $y^2 < 4x, \quad (x - 6)^2 + y^2 < 25, \quad y > 0.$

15. The equation to any straight line through the intersection of $ax + by + c = 0$ and $Ax + By + C = 0$ may be written

$$(ax + by + c) + m(Ax + By + C) = 0.$$

16. Find the equation of the straight line through (0, 1) and the intersection of $y + x = 0$ with $x - 3y + 1 = 0$.

SOLUTION OF NUMERICAL EQUATIONS.

161. Consider first the quadratic equation $x^2 - 6x - 3 = 0$. The graph of $y = x^2 - 6x - 3$ is given in the figure. It is roughly determined by the points $(-1, 4)$, $(0, -3)$, $(1, -8)$, $(2, -11)$, $(3, -12)$, $(4, -11)$, $(5, -8)$, $(6, -3)$, $(7, 4)$. The symmetry with respect to the line $x = 3$ follows from the form $y + 12 = (x - 3)^2$ in which the equation may be written. The dotted lines have the equations

$$y' \equiv y + 12 = 0, \quad x' \equiv x - 3 = 0.$$

Referred to them as new coördinate axes, the equation is $y' = x'^2$. To this form any equation of the form $y = x^2 + px + q$ may be reduced The graph is called a **parabola.**[*] To solve the equation $x^2 - 6x - 3 = 0$ consists in finding the values of x which make $y = 0$. In the graph, it consists in finding the x-coördinates of the points in which the graph crosses the x-axis. The one lies between 0 and -1, and the other between 6 and 7; the exact values are evidently $3 \pm \sqrt{12}$. It is clear that all quadratic equations in which the coefficient of x^2 has been made unity, lead to the same graph, the variation being the relative position of the axes of coördinates.

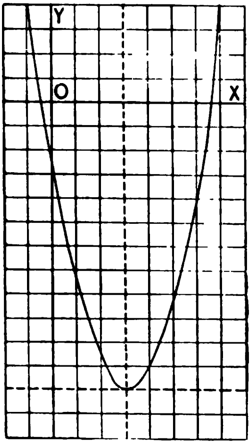

FIG. 11.

162. Consider the cubic equation $x^3 + 4x^2 - 7 = 0$. The graph of $y = x^3 + 4x^2 - 7$ is given in the figure, as sketched through the points $(-4, -7)$, $(-3\frac{1}{2}, -\frac{7}{8})$, $(-3, 2)$, $(-2\frac{1}{2}, 2\frac{3}{8})$, $(-2, 1)$, $(-1\frac{1}{2}, -1\frac{3}{8})$, $(-1, -4)$, $(-\frac{1}{2}, -6\frac{1}{8})$, $(0, -7)$,

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[*] The parabola $y^2 = 4x$, whose graph is given in § 156, may be reduced to the above form by taking $x' = y$, $y' = 4x$. This transformation turns the picture through 90° and magnifies the resulting figure in the vertical direction.

$(\frac{1}{2}, -5\frac{7}{8})$, $(1, -2)$, $(1.1, -.829)$, $(1.2, .488)$, $(1.25, 1.203)$, $(1.5, 5\frac{3}{8})$. The roots of the cubic lie between 1 and 2, between -1 and -2, and between -3 and -4. More closely, the positive root lies between 1.1 and 1.2. To determine the second decimal place, the process of direct trial of 1.11, . . . , 1.19 becomes laborious. To shorten the work, set $x = 1.1 + z$, whence the relation becomes

$$(10) \qquad y = z^3 + 7.3z^2 + 12.43z - .829.$$

In determining z to make $y = 0$, the terms z^3 and $7.3z^2$ are of little consequence, since z is of the denomination hundredths. We should expect, therefore, that z lies between .06 and .07, being given approximately by $12.43z - .829 = 0$. In fact, y is negative when $z = .06$ and is positive when $z = .07$. Setting $z = .06 + w$, the relation becomes

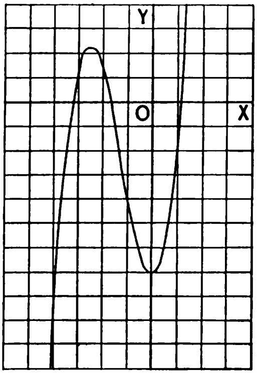

FIG. 12.

$$(11) \qquad y = w^3 + 7.48w^2 + 13.3168w - .056704.$$

Using the last two terms only, we find that $y = 0$ if $w = .0042 +$, to four decimal places. In verification, $w = .0042$ makes y negative and $w = .0043$ makes y positive. Hence the positive root of the cubic is $1.1642 +$.

163. The calculations may be shortened by Horner's method,* which is based upon the process called **Synthetic Division.**

The process of the division of $x^3 + 4x^2 - 7$ by $x + 3$ may be exhibited as follows:

$$\begin{array}{llll|l}
x^3 & + 4x^2 & + 0\cdot x & - 7 & x + 3 \quad = \text{divisor} \\
\underline{x^3} & \underline{+ 3x^2} & & & x^2 + x - 3 = \text{quotient} \\
 & x^2 & & & \\
 & \underline{x^2} & \underline{+ 3x} & & \\
 & & - 3x & & \\
 & & \underline{- 3x} & \underline{- 9} & \\
 & & & + 2 = \text{remainder.} &
\end{array}$$

---

* Published in 1819 by W. G. Horner.

Since the literal terms x^3, x^2, x may be supplied from their relative positions, we may indicate the work by **detached coefficients:**

$$\begin{array}{rrrr|l}
1 & 4 & 0 & -7 & 1\quad 3 \qquad\qquad \text{divisor} \\ \cline{5-5}
\underline{1} & \underline{3} & & & 1\quad 1\quad -3\ \text{quotient} \\
 & 1 & & & \\
 & 1 & \underline{3} & & \\
 & & -3 & & \\
 & & -3 & -9 & \\ \cline{2-4}
 & & & 2 & \text{remainder.}
\end{array}$$

We agree to omit the *first* term of each partial product (1, 1, -3, respectively) since it is merely a repetition of the number above it. Next, if the second term of the divisor be changed in sign, the new divisor being $1\ \ -3$, the sign of the *second* term of each partial product will be changed, so that the subtraction previously performed is replaced by addition. The work for the division by $x+3$ thus becomes the following:

$$\begin{array}{rrrr|l}
1 & 4 & 0 & -7 & 1\quad -3 \\ \cline{5-5}
 & \underline{-3} & & & 1\qquad 1\quad -3 \\
 & 1 & & & \\
 & & \underline{-3} & & \\
 & & -3 & & \\
 & & & \underline{+9} & \\
 & & & 2 &
\end{array}$$

Arranging the work in a horizontal row and omitting the coefficient 1 in the divisor $1\ \ -3$, the work stands thus:

$$\begin{array}{lrrrr|l}
\text{Dividend} = & 1 & 4 & 0 & -7 & -3 \\ \cline{6-6}
\text{Partial products} = & & -3 & -3 & +9 & \\ \cline{2-5}
\text{Quotient} = & 1 & 1 & -3, & 2 & = \text{remainder.}
\end{array}$$

In this final form the process is called **synthetic division.**

We next divide $3x^7+x^6-31x^4+21x-5$ by $x-2$.

$$\begin{array}{lrrrrrrrr|l}
\text{Dividend} = & 3 & 1 & 0 & -31 & 0 & 0 & 21 & -5 & +2 \\ \cline{10-10}
\text{Products} = & & 6 & 14 & 28 & -6 & -12 & -24 & -6 & \\ \cline{2-9}
\text{Quotient} = & 3 & 7 & 14 & -3 & -6 & -12 & -3, & -11 & = \text{rem.}
\end{array}$$

The quotient is $3x^6 + 7x^5 + 14x^4 - 3x^3 - 6x^2 - 12x - 3$, and the remainder is -11.

164. Returning to the example and graph of § 162, we see that one root is $x = 1 + u$, u being a positive decimal. Then

$$y = (1 + u)^3 + 4(1 + w)^2 - 7 = u^3 + 7u^2 + 11u - 2.$$

Without the expansion just made, the final form $u^3 + \ldots - 2$ may be obtained by synthetic division. If the degree of the given equation is large, say 6 or 7, the second process is much shorter than the first. Since $u = x - 1$, we have

$$y = x^3 + 4x^2 - 7; \quad y \equiv (x - 1)^3 + 7(x - 1)^2 + 11(x - 1) - 2.$$

Hence if we divide the first expression by $x - 1$, the remainder must be -2, as is the case with the second expression. The respective quotients are $x^2 + 5x + 5$ and $(x - 1)^2 + 7(x - 1) + 11$, and they must be equal. Dividing them by $x - 1$, the quotients are $x + 6$ and $(x - 1) + 7$, and the remainders are each 11. It is thus clear that in the division of $x^3 + 4x^2 - 7$ by $x - 1$ the successive remainders -2, $+11$, 7, 1 must be coefficients in reverse order of the transformed equation $u^3 + 7u^2 + 11u - 2$. By synthetic division this work may be done thus:

$$\begin{array}{rrrrl}
1 & 4 & 0 & -7 & \underline{|\;1\;} \\
 & 1 & 5 & 5 & \\ \hline
1 & 5 & 5, & -2 & = \text{first remainder} \\
 & 1 & 6 & & \\ \cline{1-3}
1 & 6, & 11 & = & \text{second remainder} \\
 & 1 & & & \\ \cline{1-2}
1, & 7 & = & & \text{third remainder.}
\end{array}$$

The general theorem involved is stated in the next section.

165. *Given* $y = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \ldots + p_{n-1}x + p_n$ *and* $x = u + h$, *to obtain the expression for* y *in terms of* u, *we divide* $p_0x^n + \ldots + p_n$ *by* $x - h$ *and call the quotient* Q_1 *and the remainder*

r_1, then divide Q_1 by $x - h$ and call the quotient Q_2 and the remainder r_2, etc. The required expression for y is

$$p_0u^n + r_nu^{n-1} + \dots + r_3u^2 + r_2u + r_1.$$

In proof, we observe that the final expression equals

$$E \equiv p_0(x-h)^n + r_n(x-h)^{n-1} + \dots + r_3(x-h)^2 + r_2(x-h) + r_1.$$

To prove that E is another form of the initial function $y = p_0x^n + \dots + p_n$, we divide both E and y by $x - h$. By inspection, the quotient of E is

$$p_0(x-h)^{n-1} + r_n(x-h)^{n-2} + \dots + r_3(x-h) + r_2$$

and the remainder r_1. By the notations of the theorem, the division of $y \equiv p_0x^n + \dots + p_n$ by $x - h$ yields the quotient Q_1 and the remainder r_1. Hence will $E = y$ *if*

$$Q_1 = p_0(x-h)^{n-1} + r_n(x-h)^{n-2} + \dots + r_3(x-h) + r_2.$$

Dividing each member by $x - h$, the quotients are respectively

$$(12) \qquad Q_2, \quad p_0(x-h)^{n-2} + r_n(x-h)^{n-3} + \dots + r_3$$

and the remainder is r_2 in each case. Hence will $E = y$ *if* the expressions (12) are equal. After $n - 1$ such divisions, the quotients are Q_{n-1} and $p_0(x-h) + r_n$, and E will equal y *if* these quotients are equal. Dividing them by $x - h$, we obtain in both cases the quotient p_0 and the remainder r_n. Hence $E \equiv y$.

A more direct proof follows from the remainder theorem (§ 27). Assume that $y \equiv E$, where the coefficients $r_n, \dots, r_2, r_1$ of E are as yet undetermined. Setting $x = h$, we get

$$p_0h^n + p_1h^{n-1} + \dots + p_{n-1}h + p_n = r_1.$$

But the left member is the value of y when h is written for x and is therefore the remainder left on dividing y by $x - h$ (§ 27). By inspection, the right member r_1 is the remainder left on dividing E by $x - h$. Since the two remainders are equal and the two dividends y and E are assumed to be equal, the quotients must be equal:

$$Q_1 \equiv p_0(x-h)^{n-1} + r_n(x-h)^{n-2} + \dots + r_3(x-h) + r_2.$$

Proceeding as before, the division of Q_1 by $x - h$ leaves a remainder equal to r_2; etc. Hence the undetermined coefficients $r_n, \ldots, r_2, r_1$ in E turn out to have the values given in the theorem.

166. Resuming the equation $y = u^3 + 7u^2 + 11u - 2$ of § 164, where u is a decimal, we seek the value of u which makes $y = 0$. The powers u^3 and u^2 are small relative to $11u - 2$. Hence u is approximately .1. In verification, we observe that y is negative for $u = .1$ and positive for $u = .2$ and hence is zero for some intermediate value (by the graph or by the formal proof in § 170). Setting $u = .1 + z$, the equation for z is found by synthetic division (§ 165) as follows:

$$\begin{array}{rrrrl}
1 & 7 & 11 & -2 & \underline{|.1} \\
 & .1 & .71 & 1.171 & \\
\cline{1-4}
1 & 7.1 & 11.71\,, & -\;\; .829 & = \text{first remainder } r_1 \\
 & .1 & .72 & & \\
\cline{1-3}
1 & 7.2\,, & 12.43 & \multicolumn{2}{l}{= \text{second remainder } r_2} \\
 & .1 & & & \\
\cline{1-2}
1\,, & 7.3 & \multicolumn{3}{l}{= \text{third remainder } r_3.}
\end{array}$$

$\therefore \quad y = z^3 + 7.3z^2 + 12.43z - .829$, a result identical with (10). Hence, as in § 162, $z = .06 + w$, where w is of the denomination hundredths. By synthetic division, we form the equation (11) for w:

$$\begin{array}{rrrrl}
1 & 7.3 & 12.43 & -\;.829 & \underline{|.06} \\
 & .06 & .4416 & .772296 & \\
\cline{1-4}
1 & 7.36 & 12.8716\,, & -\;.056704 & \\
 & .06 & .4452 & & \\
\cline{1-3}
1 & 7.42\,, & 13.3168 & & \\
 & .06 & & & \\
\cline{1-2}
1\,, & 7.48 & & &
\end{array}$$

$$\therefore \quad y = w^3 + 7.48w^2 + 13.3168w - .056704.$$

Then $y = 0$ for $w = .0042 +$, so that the positive root is $1.1642 +$.

167. For a negative root $-x$ of $x^3 + 4x^2 - 7 = 0$, we may solve the corresponding equation $x^3 - 4x^2 + 7 = 0$ with the root $+x$, or we may proceed with the given equation by the usual method. Thus for the root lying between -3 and -4, we set $x = -4 + t$, where t is positive and of the denomination tenths and proceed as usual. The work is exhibited compactly as follows:

$$\begin{array}{rrrrl}
1 & 4 & 0 & -7 & \underline{|-4} \\
 & -4 & 0 & 0 & \\
\cline{1-4}
1 & 0 & 0 & ,\ -7 & |\underline{.6} \\
 & -4 & 16 & & \\
\cline{1-3}
1 & -4 & ,\ 16 & & \\
 & -4 & & & \\
\cline{1-2}
1, & -8 & & & \\
 & .6 & -4.44 & 6.936 & \\
\cline{1-4}
1 & -7.4 & 11.56 & ,\ -\ .064 & |\underline{.008} \\
 & .6 & -4.08 & & \\
\cline{1-3}
1 & -6.8\ , & 7.48 & & \\
 & .6 & & & \\
\cline{1-2}
1, & -6.2 & & & \\
 & .008 & -\ .049536 & .0594437 & \\
\cline{1-4}
1 & -6.192 & 7.430464\ , & -\ .0045563 & |\underline{.000617} \\
 & .008 & -\ .049472 & & \\
\cline{1-3}
1 & -6.184\ , & 7.380992 & & \\
 & .008 & & & \\
\cline{1-2}
1, & -6.176 & & &
\end{array}$$

Hence, to six decimal places, the root is $-4+.608617=-3.391383$.

It is seen that the final terms $16t - 7$ of the first transformed equation do not suffice to determine t very accurately; whereas in the later transformed equations the last two terms determine the figure of the root sought.

Similarly, the third root is -1.7728, to four decimal places.

168. Consider as a new example the equation $y=x^3+6x^2+8x+8$, whose graph is given in Fig. 13, having been traced through the points $(1, 23)$, $(.5, 13\frac{5}{8})$, $(0, 8)$, $(-.5, 5\frac{3}{8})$, $(-1, 5)$, $(-1.5, 6\frac{1}{8})$, $(-2, 8)$, $(-3, 11)$, $(-4, 8)$, $(-4.5, 2\frac{3}{8})$, $(-5, -7)$. By accurate plotting in the vicinity of the two bends in the curve, we find that they occur at $x = -.845$ and $x = -3.155$, approximately.* The corresponding y-coördinates are 4.9209 and 11.079, respectively. The fact that the lowest point of the bend which points downwards is above the x-axis follows more simply from the form $y = (x + 2)^3 - 4x$ in which the equation may be written. Hence, for $-2 < x < 0$, $x + 2$ and $-4x$ are positive and therefore also y. On account of the double bend, the graph crosses the x-axis but once; for, if the branch below the x-axis should bend and cross the axis again (necessarily at the left of $x = -5$), or if the upper branch should bend and cross the x-axis (necessarily at the right of $x = +1$), it would be possible to draw a straight line which would meet the graph in at least four points. To show that this is impossible, we note that the equation of the straight line has the form $y = mx + n$. The x-coördinates of its intersections with the curve are given by

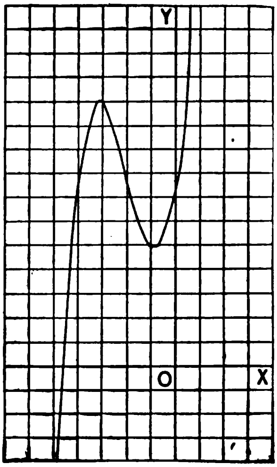

FIG. 13.

$$mx + n = x^3 + 6x^2 + 8x + 8.$$

By § 77, this equation of the third degree has at most three roots.

To give an algebraic proof, we observe that, for $x < -6$, $x + 2 < -4$, so that the condition $(x + 2)^3 < 4x$ will be satisfied if $-(x + 2)^2 < x$, viz., if $-(x + \frac{5}{2})^2 < -(\frac{3}{2})^2$, which is true. If $x > 2$, then $x + 2 > 4$, and the condition $(x + 2)^2 > 4x$ will be satisfied if $(x + 2)^2 > x$, viz., if $(x + \frac{3}{2})^2 > \frac{1}{4}$, which is true.

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* By the Calculus, the values of x are the roots $-2 \pm \frac{1}{3}\sqrt{12}$ of the equation $3x^2 + 12x + 8 = 0$.

169. Descartes' Rule of Signs.* *An equation $f(x) = 0$ with real coefficients cannot have more positive roots than there are changes of sign in $f(x)$, and cannot have more negative roots than there are changes of sign in $f(-x)$.*

For example, the terms of $f(x) \equiv + x^3 + 6x^2 + 8x + 8 = 0$ all have $+$ signs, so that there are no positive roots; there are three changes of sign in $f(-x) \equiv - x^3 + 6x^2 - 8x + 8 = 0$, so that there are not more than three negative roots.

The equation $f(x) \equiv x^9 - 5x^8 - x^3 + 7x - 2 = 0$ has at most three positive roots, and at most two negative roots since

$$f(-x) \equiv - x^9 - 5x^8 + x^3 - 7x - 2 = 0$$

has two changes of sign. Hence the equation must have at least $9 - 5 = 4$ imaginary roots [see § 181].

Suppose, in general, that the positive roots of $f(x) = 0$ are $a_1, a_2, \ldots, a_p$. Then $f(x)$ is divisible (§ 27) by the product

$$(x - a_1)(x - a_2) \ldots (x - a_p).$$

If the quotient is $Q(x)$, the roots of $Q(x) = 0$ give all the negative and imaginary roots of $f(x) = 0$. Let the signs of the terms of $Q(x)$ be, for example,

$$+ + - - + - - - + - +.$$

To find the signs of the product $(x - a_1)Q(x)$, we employ as multiplier a binomial with the signs $+ -$. The signs of the products are therefore

$$\begin{array}{lcccccccccccc}
 & + & + & - & - & + & - & - & - & + & - & + & \\
 & & - & - & + & + & - & + & + & + & - & + & - \\
\hline
\text{Sum:} & + & \pm & - & \mp & + & - & \mp & \mp & + & - & + & -
\end{array}$$

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* Descartes' La Géométrie, Leyden, 1637. For an account of his life and work, see Ball, History of Mathematics, pp. 270–279.

We observe that in the product an ambiguity of sign replaces each continuation of sign in $Q(x)$, the signs before and after an ambiguity or set of ambiguities are different, and a change of sign is introduced at the end. Hence for every isolated ambiguity and the sign just preceding and just following it, the number of changes of sign (whether we take the upper or lower sign in the ambiguity) is the same as for the three corresponding signs in $Q(x)$. An analogous result holds for a set of ambiguities if we take all the ambiguities in the set to be alike. But if the latter are not taken alike, there is one or more additional changes of sign in the product than in $Q(x)$. There is always an additional change of sign at the end. Hence the product contains at least one more change of sign than $Q(x)$. After multiplying in the p factors $x - a_1, x - a_2, \ldots, x - a_p$, we obtain $f(x)$, which therefore has at least p more changes of sign than $Q(x)$. Hence $f(x)$ has at least p changes of sign. The number of positive roots is thus $\overset{=}{<} c$, c being the actual number of changes of sign in $f(x)$.

The negative roots of $f(x) = 0$ are the positive roots of $f(-x) = 0$, so that the number of the former is less than or equal to the number of changes of sign in $f(-x)$.

Thus $x^6 - 3x^2 + x + 1 = 0$ has at most two positive roots and at most two negative roots and consequently has at least two imaginary roots.

170. Consider the graph of $y = f(x)$, where $f(x)$ is a rational integral function of x (§ 76). If there be two points of the graph, one of which is above the x-axis and the other below, it may be proved that the graph must cross the x-axis at one or more intermediate points. Thus if $f(a)$ is positive and $f(b)$ is negative, there is some value of x intermediate to a and b for which $f(x)$ is zero. The proof consists in showing that, as x changes gradually from a to b, $f(x)$ also changes gradually and assumes all intermediate values between $f(a)$ and $f(b)$, so that in passing from a positive to a negative value it passes through the value zero. While this is readily proved (see example below) when $f(x)$ is of the form

$$x^n + p_1x^{n-1} + \dots + p_n,$$

the case under consideration, it is not true for all functions $f(x)$. For example, it fails for the functions $y = 2\sqrt{x}$ and $y = \sqrt{10x - x^2}$ whose graphs were constructed in § 156, since there are two values of y corresponding to each value of x.

Consider the example $f(x) = x^3 + 4x^2 - 7$. As x changes from a to $a + h$, $f(x)$ changes from $f(a)$ to $f(a + h)$, the increase being

$$\{(a+h)^3 + 4(a+h)^2 - 7\} - \{a^3 + 4a^2 - 7\}$$
$$= (3a^2 + 8a)h + (3a + 4)h^2 + h^3.$$

If h is small, this difference is small, and by taking h sufficiently small, this difference can be made as small as we please. Hence if x changes by small amounts from a to b, $f(x)$ will change by small amounts from $f(a)$ to $f(b)$.

Moreover, a curve joining two points which lie on opposite sides of the x-axis must cross that line an *odd* number of times; but must cross it an *even* number of times, or not at all, when the points lie on the same side of the x-axis. We may state the theorem:

If $f(x)$ be a rational integral function with real coefficients, and the values $f(a)$, $f(b)$ given by substituting real numbers a, b for x are of opposite signs, the equation $f(x) = 0$ has an odd number of real roots lying between a and b; but has an even number of real roots or no real root lying between a and b, if $f(a)$ and $f(b)$ are of like sign.

EXAMPLE 1. Every equation of *odd* degree has at least one real root of sign opposite to that of the constant term.

Thus $x^3 + 4x^2 - 7 = 0$ was seen to have a root between $+1$ and $+2$.

Let the equation be $f(x) \equiv x^n + ax^{n-1} + \dots + c = 0$, and let x approach the values $-\infty$, 0, and $+\infty$. The sign of $f(0)$ is the same as that of c. To find the sign of $f(\pm\infty)$, divide $f(x)$ by x^{n-1}, which is positive since $n - 1$ is even. The quotient $x + a + \frac{b}{x} + \dots + \frac{c}{x^{n-1}}$ differs from x by a function which approaches zero as x approaches $\pm\infty$. Hence $f(+\infty)$ is positive and

$f(-\infty)$ is negative. Hence, if c is positive, $f(x)$ changes its sign between $x = -\infty$ and $x = 0$ and therefore $f(x) = 0$ has a negative root. If c is negative, $f(x)$ changes its sign between $x = 0$ and $x = +\infty$, so that $f(x) = 0$ has a positive root.

EXAMPLE 2. Every equation of *even* degree, whose constant term is negative, has at least one positive and at least one negative root.

Upon substituting $-\infty$, 0, $+\infty$ for x, the results are $+\infty$, c, $+\infty$, respectively, where c is negative. Hence there is a real root between $-\infty$ and 0, and a real root between 0 and $+\infty$.

EXERCISES.

1. Plot the graph of $y = x^2 + 2x - 1$ and solve $x^2 + 2x - 1 = 0$.

2. Plot $y = x^3 + 2x + 20$ and find to five decimals the root of $x^3 + 2x + 20 = 0$ such that $-3 < x < -2$.

3. Plot $y = x^3 + x - 3$ and find to five decimals the root $1 < x < 2$.

4. Plot $y = x^3 - 4x$. What are the roots of $x^3 - 4x = 0$?

5. Show that the graph of $y = x^3 + 3x^2 + 2x + 1$ crosses the x-axis but once.

6. Find the roots between 1 and 2, -1 and -2, of $x^3 + 3x^2 - 2x - 5 = 0$.

7. Find the root between 2 and 3 of $x^3 + 3x^2 - 3x - 18 = 0$.

8. Find the real roots of $x^3 + 2x^2 - x - 1 = 0$.

9. Find the real roots of $x^4 + x^3 - x - 2 = 0$.

10. Find the real roots of $x^4 + 4x^3 + 6x^2 - 8x - 4 = 0$.

11. If the terms of an equation are all positive, it has no positive root.

12. If the terms of an equation, with all powers of x present, are alternately positive and negative, it has no negative root.

13. The equation $x^3 + a^2x + b^2 = 0$ has one negative root and two imaginary roots.

14. If n is even, $x^n + 1 = 0$ has no real root; if n is odd, -1 is the only real root.

15. The equation $x^8 + 2x^3 + 3x - 1 = 0$ has at least six imaginary roots.

16. $x^4 + 12x^2 + 5x - 9 = 0$ has a positive root, a negative root, and two imaginary roots.

17. For n even, $x^n - 1 = 0$ has only two real roots; for n odd, only one.

18. $x^6 - 7x^4 - 3x^2 + 7 = 0$ has at least two imaginary roots.

CHAPTER XIX.

THEORY OF EQUATIONS.

171. The solution of the general quadratic equation (see § 31) was known to the Arabians as early as the ninth century. The solution of the cubic equation $x^3 + qx + r = 0$ was discovered by Scipio Ferreo, who imparted it to his pupil Florido, in 1505. The latter proclaimed his knowledge of the solution upon learning that Tartaglia had solved the analogous equation $x^3 + mx^2 + n = 0$. But Tartaglia doubted the truth of his statement and challenged him to a trial in the year 1535, having succeeded in rediscovering Ferreo's solution. Then Cardan appeared and solicited from Tartaglia his rules for solving various cubic equations and finally succeeded in getting them after giving the most sacred promises of secrecy. But in 1545 Cardan broke his promises and published the rules in his *Ars Magna*. Tartaglia had intended to publish his rules in his own work, but died in 1559 before the part on cubic equations was reached, so that no mention is made by him of his discoveries. In time the rules came to be regarded as the discovery of Cardan and to bear his name.

172. Consider the cubic equation in the most general form

$$\text{(G)} \qquad x^3 + ax^2 + bx + c = 0.$$

Setting $x = y - \frac{a}{3}$, the equation becomes

$$\text{(R)} \qquad y^3 + \left(b - \frac{a^2}{3}\right)y + \left(c - \frac{ab}{3} + \frac{2a^3}{27}\right) = 0,$$

an equation lacking the square of the unknown quantity, and called the **reduced form** of equation (G). It suffices to solve the equation in y, since the values of x are then given by the relation $x = y - a/3$.

Consider then the **reduced cubic equation**

$$y^3 + py + r = 0. \tag{1}$$

Making the substitution

$$y = z - \frac{p}{3z}, \tag{2}$$

the cubic (1) becomes

$$z^3 - \frac{p^3}{27z^3} + r = 0.$$

Multiplying by z^3, the latter becomes

$$z^6 + rz^3 - \frac{p^3}{27} = 0. \tag{3}$$

Considering z^3 to be the unknown quantity, equation (3) is a quadratic equation, the solution of which gives z^3 and then

$$z = \sqrt[3]{-\frac{r}{2} \pm \sqrt{R}}, \qquad R \equiv \frac{r^2}{4} + \frac{p^3}{27}. \tag{4}$$

Remembering that a quantity q has three cube roots* (§ 8),

$$\sqrt[3]{q,}\,|\quad \omega\sqrt[3]{q,}\,|\quad \omega^2\sqrt[3]{q}\,|\qquad (\omega = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3},\ \omega^3 = 1)$$

it would seem that the six values of z, given by (4), would lead to six roots y, given by (2), of the cubic equation (1). But

$$\left(-\frac{r}{2} + \sqrt{R}\right)\left(-\frac{r}{2} - \sqrt{R}\right) = \frac{r^2}{4} - R = -\frac{p^3}{27}.$$

Hence, if the notation of two of the cube roots (4) be definitely fixed in such a manner that

$$\sqrt[3]{-\frac{r}{2} + \sqrt{R}}\,\Big| \cdot \sqrt[3]{-\frac{r}{2} - \sqrt{R}}\,\Big| = -\frac{p}{3},$$

---

* The vertical bar at the right of the radical sign is used to mark the choice of a particular root.

then the remaining four cube roots (4) may be paired so that the product of the two in each pair is $-p/3$, viz.,

$$\omega\sqrt[3]{-\frac{r}{2}+\sqrt{R}}\cdot\omega^2\sqrt[3]{-\frac{r}{2}-\sqrt{R}}=-\frac{p}{3},$$

$$\omega^2\sqrt[3]{-\frac{r}{2}+\sqrt{R}}\cdot\omega\sqrt[3]{-\frac{r}{2}-\sqrt{R}}=-\frac{p}{3}.$$

But the value of y given by (2) remains unaltered if we replace z by $\frac{-p}{3z}$; indeed,

$$-\frac{p}{3z}-\frac{p}{3}\div\left(\frac{-p}{3z}\right)=z-\frac{p}{3z}.$$

Hence the two values of z in any one of the three pairs give the same value to y, and the value of y equals the sum of the two values of z in that pair. Hence the *three* roots y are

$$(5)\qquad\left\{\begin{aligned} y_1&=\sqrt[3]{\frac{-r}{2}+\sqrt{R}}+\sqrt[3]{\frac{-r}{2}-\sqrt{R}},\\ y_2&=\omega\sqrt[3]{\frac{-r}{2}+\sqrt{R}}+\omega^2\sqrt[3]{\frac{-r}{2}-\sqrt{R}},\\ y_3&=\omega^2\sqrt[3]{\frac{-r}{2}+\sqrt{R}}+\omega\sqrt[3]{\frac{-r}{2}-\sqrt{R}}.\end{aligned}\right.$$

EXAMPLE. Solve the reduced cubic equation $y^3-15y-126=0$.

Set $y=z-\left(\frac{-15}{3z}\right)=z+\frac{5}{z}$. Then $z^3-126z^3+125=0$.

$$\therefore\quad (z^3-1)(z^3-125)=0.$$

$\therefore\ z=1,\ \omega,\ \omega^2,\ 5,\ 5\omega$, or $5\omega^2$. Then $y=z+\frac{5}{z}$ equals respectively

$$6,\quad \omega+\frac{5}{\omega}=\omega+5\omega^2,\quad \omega^2+5\omega,\quad 6,\quad 5\omega+\omega^2,\quad 5\omega^2+\omega,$$

giving only three distinct roots y. Since $\omega^2 + \omega + 1 = 0$ (§ 8), they are 6, $-4\omega - 5$, $4\omega - 1$. Since $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-3}$, they become

$$6, \quad -3-2\sqrt{-3}, \quad -3+2\sqrt{-3}.$$

The two imaginary roots are *conjugate* (§ 7).

173. Formulæ (5) for the roots are known as the formulæ of Cardan. The method used for obtaining them is essentially that given by Hudde in 1650.

According to the value of $R \equiv \frac{r^2}{4} + \frac{p^3}{27}$, we have, when p and r are real, *for R positive, one real root and two conjugate imaginary roots; for R equal zero, all roots real and two of them equal; for R negative, all three roots real and distinct.*

In proof, suppose first that $R > 0$. Let the radicals be A and B so that the roots are $A + B$, $\omega A + \omega^2 B$, $\omega^2 A + \omega B$. Since

$$\omega = -\tfrac{1}{2} + \tfrac{1}{2}\sqrt{-3}, \qquad \omega^2 = -\tfrac{1}{2} - \tfrac{1}{2}\sqrt{-3},$$

the three roots are $A + B$ and $-\frac{1}{2}(A + B) \pm \frac{1}{2}(A - B)\sqrt{-3}$. Since A and B are now real, one root is real and the others are conjugate imaginaries.

If $R = 0$, the roots are $A + B, -\frac{1}{2}(A + B)$, $-\frac{1}{2}(A + B)$, and hence are all real.

If $R < 0$, the radicals in formula (5) are imaginary. But (see Appendix) there exists a cube root of $-\frac{r}{2} + \sqrt{R}$ of the form $\alpha + \beta\sqrt{-1}$, α and β being real, the other two cube roots being $\omega(\alpha + \beta\sqrt{-1})$ and $\omega^2(\alpha + \beta\sqrt{-1})$. The conjugate imaginary $-\frac{r}{2} - \sqrt{R}$ will then have as its cube roots $\alpha - \beta\sqrt{-1}$, $\omega^2(\alpha - \beta\sqrt{-1})$, $\omega(\alpha - \beta\sqrt{-1})$. Hence the roots (5) may be written (since the product of the two terms in any root is the real number $-p/3$):

$$(\alpha+\beta\sqrt{-1})+(\alpha-\beta\sqrt{-1})=2\alpha,$$

$$\omega(\alpha+\beta\sqrt{-1})+\omega^2(\alpha-\beta\sqrt{-1})=-\alpha-\beta\sqrt{3},$$

$$\omega^2(\alpha+\beta\sqrt{-1})+\omega(\alpha-\beta\sqrt{-1})=-\alpha+\beta\sqrt{3}.$$

These real roots are readily seen to be unequal.

Thus when the roots of a cubic equation are all real and unequal, Cardan's formulæ present their values in a form involving imaginaries. In eliminating these imaginaries, it was necessary to express the cube root of a complex number in the form of $\alpha+\beta\sqrt{-1}$. If we attempt to do this algebraically, we find that α, β depend upon equations similar to the proposed equation. For example, α depends upon a cubic equation all of whose roots are real, so that its roots would depend upon expressions as complicated as α itself. Hence the case when all three roots are real is called the "irreducible case," since Cardan's formulæ then have no practical value. But this case is precisely the one which can be readily solved by trigonometry. [See Appendix.]

174. Since $\omega^2+\omega+1=0$, we find by formulæ (5) that

$$y_1+y_2+y_3=(1+\omega+\omega^2)(A+B)=0,$$

$$y_1y_2+y_1y_3+y_2y_3=(3\omega^2+3\omega)AB=-3AB=-3\left(\frac{-p}{3}\right)=p,$$

$$y_1y_2y_3=(A+B)(A^2+B^2+\omega^2AB+\omega AB)=A^3+B^3=-r,$$

where we have set $A\equiv\sqrt[3]{\frac{-r}{2}+\sqrt{R}}$, $B\equiv\sqrt[3]{\frac{-r}{2}-\sqrt{R}}$.

Hence in a cubic equation lacking the term y^2, the sum of the roots is zero, the sum of the products of the roots taken two at a time is the coefficient of y, and the product of all the roots is the negative of the constant term.

EXERCISES.

Solve the cubic equations

1. $x^3 - 18x + 35 = 0$.
2. $x^3 - 15x + 4 = 0$.
3. $x^3 - 27x + 54 = 0$.
4. $x^3 + 6x^2 + 3x + 18 = 0$.
5. $28x^3 + 9x^2 - 1 = 0$.
6. $x^3 + 4x^2 - 11x + 6 = 0$.
7. Since $x_1 = y_1 - a/3$, $x_2 = y_2 - a/3$, $x_3 = y_3 - a/3$ are the roots of the general equation (G) if y_1, y_2, y_3 are the roots of the reduced equation (R), show that $x_1 + x_2 + x_3 = -a$, $x_1x_2 + x_1x_3 + x_2x_3 = b$, $x_1x_2x_3 = -c$.

175. The solution of equations of the fourth degree was attempted by Cardan, but was first accomplished by his pupil Ferrari. It is sometimes ascribed to Bombelli, who published it in his algebra in 1579. Other solutions have been given by Descartes in 1637 and by Euler in 1770 and by many later writers.

176. *Ferrari's solution of the quartic equation*

$$x^4 + px^3 + qx^2 + rx + s = 0. \tag{6}$$

The equation may be written

$$\left(x^2 + \frac{p}{2}x\right)^2 = \left(\frac{p^2}{4} - q\right)x^2 - rx - s.$$

In case the second member is not a perfect square, add to each member $\left(x^2 + \frac{p}{2}x\right)y + \frac{y^2}{4}$. Then

$$\left(x^2 + \frac{p}{2}x + \frac{y}{2}\right)^2 = \left(y + \frac{p^2}{4} - q\right)x + \left(\frac{py}{2} - r\right)x + \left(\frac{y^2}{4} - s\right). \tag{7}$$

The second member will be a perfect square (§ 31) if

$$\left(y + \frac{p^2}{4} - q\right)\left(y^2 - 4s\right) - \left(\frac{py}{2} - r\right)^2 = 0.$$

Hence y must be a root of the cubic equation

$$y^3 - qy^2 + (pr - 4s)y + 4qs - p^2s - r^2 = 0. \tag{8}$$

Let y_1 be a root of (8) and set $y_1 + \frac{p^2}{4} - q \equiv T$. The right member of (7) may be written

$$T\left[x+\frac{1}{2}\left(\frac{py_1}{2}-r\right)\Big/T\right]^2.$$

Hence (7) factors and gives

$$(9)\qquad \begin{cases} x^2+\left(\frac{p}{2}+\sqrt{T}\right)x+\frac{y_1}{2}+\frac{1}{2}\left(\frac{py_1}{2}-r\right)\div\sqrt{T}=0,\\ x^2+\left(\frac{p}{2}-\sqrt{T}\right)x+\frac{y_1}{2}-\frac{1}{2}\left(\frac{py_1}{2}-r\right)\div\sqrt{T}=0. \end{cases}$$

Solving these quadratics, we obtain the roots $x_1, x_2; x_3, x_4$ of (6) We observe (§ 32) that

$$x_1+x_2=-\frac{p}{2}-\sqrt{T},\quad x_1x_2=\frac{y_1}{2}+\frac{1}{2}\left(\frac{py_1}{2}-r\right)\div\sqrt{T},$$

$$x_3+x_4=-\frac{p}{2}+\sqrt{T},\quad x_3x_4=\frac{y_1}{2}-\frac{1}{2}\left(\frac{py_1}{2}-r\right)\div\sqrt{T}.$$

Hence

$$x_1+x_2+x_3+x_4=-p,\quad x_1x_2x_3x_4=\frac{1}{4}\left\{y_1^2-\left(\frac{py}{2}-r\right)^2\div T\right\}=s,$$

$$x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4=x_1x_2+(x_1+x_2)(x_3+x_4)+x_3x_4=q,$$

$$x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4=x_1x_2(x_3+x_4)+x_3x_4(x_1+x_2)=-r.$$

For a general quartic equation (4), *the sum of the roots is* $-p$, *the sum of the products of the roots two at a time is* q, *the sum of the products three at a time is* $-r$, *the product of the roots is* s.

EXAMPLE. Solve the quartic equation $x^4+2x^3-12x^2-10x+3=0$. The auxiliary cubic (8) is here $y^3+12y^2-32y-256=0$, one of whose roots is -4. Taking $y_1=-4$, we get $T=9$. Hence equations (9) are

$$x^2+4x-1=0,\qquad x^2-2x-3=0,$$

whose roots are $-2\pm\sqrt{5}$ and $3, -1$.

177. If we set $x=z-\dfrac{p}{4}$ in (6), we obtain for z a quartic equation lacking the term z^3. Consider then the **reduced form**

$$(10)\qquad z^4+Qz^2+Rz+S=0.$$

Descartes' solution consists in factoring the left member of (10) into

$$(z^2+kz+l)(z^2-kz+m) \equiv z^4+(l+m-k^2)z^2+k(m-l)z+lm.$$

In order that this expression shall be identical with the left member of (10), the corresponding coefficients must be equal:

$$l+m-k^2=Q, \quad k(m-l)=R, \quad lm=S.$$

Substituting the values of l and m derived from the first two relations into the third relation, we get

$$(Q+k^2+R/k)(Q+k^2-R/k)=4S.$$

$$\therefore \quad k^6+2Qk^4+(Q^2-4S)k^2-R^2=0.$$

We have thus a cubic equation for k^2. When it is solved, we derive the corresponding values of l and m. Then the roots of (10) are given by the solution of the two quadratic equations

$$z^2+kz+l=0, \quad z^2-kz+m=0.$$

For example, to solve $z^4-3z^2+6z-2=0$, we have the conditions

$$l+m-k^2=-3, \quad k(m-l)=6, \quad lm=-2.$$

$$\therefore \quad k^6-6k^4+17k^2-36=0.$$

The latter is satisfied by $k^2=4$. Taking $k=2$, we get $m=2$, $l=-1$.

$$\therefore \quad z^4-3z^2+6z-2=(z^2+2z-1)(z^2-2z+2).$$

The roots are therefore $-1 \pm \sqrt{2}$ and $1 \pm \sqrt{-1}$.

178. *Euler's solution* of the quartic equation (10) consists in assuming as the general expression for a root

$$z=\sqrt{u}+\sqrt{v}+\sqrt{w}.$$

Squaring and transposing the terms free of radicals, we get

$$z^2-u-v-w=2(\sqrt{u}\sqrt{v}+\sqrt{u}\sqrt{w}+\sqrt{v}\sqrt{w}).$$

Squaring the latter and reducing, we get

$$z^4-2(u+v+w)z^2-8\left(u\sqrt{v}\sqrt{w}+v\sqrt{u}\sqrt{w}+w\sqrt{u}\sqrt{v}\right)+A=0,$$

where $\qquad A\equiv(u+v+w)^2-4(uv+uw+vw).$

$$\therefore\quad z^4-2(u+v+w)z^2-8z\sqrt{u}\sqrt{v}\sqrt{w}+A=0.$$

Comparing the latter with equation (10), we get

$$u+v+w=-\tfrac{1}{2}Q,\quad \sqrt{u}\sqrt{v}\sqrt{w}=-\tfrac{1}{8}R,\quad A=S.$$

$$\therefore\quad uvw=\tfrac{1}{64}R^2,\quad uv+uw+vw=\tfrac{1}{16}Q^2-\tfrac{1}{4}S.$$

Hence, by Exercise 7 of § 174 or by the theorem of § 182, u, v, w are the roots of the cubic equation

$$t^3+\tfrac{1}{2}Qt^2+\left(\tfrac{1}{16}Q^2-\tfrac{1}{4}S\right)t-\tfrac{1}{64}R^2=0.$$

Since the expression for z involves three square roots, it might seem that the quartic equation would have eight roots. But the signs of the square roots must be chosen so that $\sqrt{u}\sqrt{v}\sqrt{w}=-\frac{1}{8}R$, so that only two signs are arbitrary, giving four roots.

EXERCISES.

Solve the quartic equations

1. $z^4-2z^2-8z-3=0$. 2. $x^4-2x^3-7x^2+8x+12=0$.

3. $z^4-10z^2-20z-16=0$. 4. $x^4-8x^3+9x^2+8x-10=0$.

5. If a quartic has two equal roots, the auxiliary cubic has two equal roots, and inversely.

6. If a quartic has two distinct pairs of equal roots, then two of the roots of Euler's auxiliary cubic are zero.

7. The roots of the quartic (6) are $\frac{1}{4}\left(-p\pm\sqrt{A}\pm\sqrt{B}\pm\sqrt{C}\right)$, all the signs being +, or two signs − and one sign +, if A, B, C are the roots of

$$W^3+(8q-3p^2)W^2+(3p^4-16p^2q+16q^2+16pr-64s)W-(p^3-4pq+8r)^2=0.$$

8. Solve the general cubic by multiplying in a new factor $x-\lambda$, and using λ to make the auxiliary cubic solvable by inspection.

179. During the eighteenth century many unsuccessful attempts were made to solve the general equations of the fifth and higher degrees. In 1770 Lagrange analyzed the methods of his predecessors and traced all their results to one principle and proved that the general quintic equation cannot be solved in this way, since the auxiliary equation is of the sixth degree and of essentially general character. It was then proved * by Abel, Wantzel, and Galois that it is impossible to solve by algebraic means the general equation of degree n for $n > 4$. But the equation of the fifth degree may be solved in terms of elliptic functions † (Hermite). The roots of certain equations of the fifth degree may be expressed by radicals. ‡

EXAMPLE 1. Find all the roots of $x^5 - 1 = 0$.

The four roots different from the root 1 satisfy

$$\frac{x^5 - 1}{x - 1} = x^4 + x^3 + x^2 + x + 1 = 0.$$

If we divide by x^2 and set $x + \frac{1}{x} = y$, we find that

$$y^2 + y - 1 = 0, \quad y = \tfrac{1}{2}(-1 \pm \sqrt{5}).$$

$$\therefore \quad x^2 - \frac{x}{2}(-1 \pm \sqrt{5}) + 1 = 0.$$

Hence there are four imaginary fifth roots of unity :

$$\tfrac{1}{4}\{-1 + \sqrt{5} \pm \sqrt{10 + 2\sqrt{5}}\ \sqrt{-1}\}, \quad \tfrac{1}{4}\{-1 - \sqrt{5} \pm \sqrt{10 - 2\sqrt{5}}\ \sqrt{-1}\}$$

If we denote one of them by ϵ, the others are ϵ^2, ϵ^3, ϵ^4. Indeed, since $\epsilon^5 = 1$, then $(\epsilon^2)^5 = 1$, $(\epsilon^3)^5 = 1$, $(\epsilon^4)^5 = 1$. Also no two of them are equal; for example, $\epsilon = \epsilon^3$ gives $\epsilon = \pm 1$, $\epsilon = \epsilon^4$ gives $\epsilon^2 = \epsilon^5 = 1$. We notice that

$$1 + \epsilon + \epsilon^2 + \epsilon^3 + \epsilon^4 = 0, \quad 1 \cdot \epsilon \cdot \epsilon^2 \cdot \epsilon^3 \cdot \epsilon^4 = \epsilon^{10} = 1.$$

EXAMPLE 2. Solve the equation

$$y^5 + py^3 + \tfrac{1}{5}p^2y + r = 0.$$

Substituting

(11) $$y = z - \frac{p}{5z},$$

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* See Serret, *Cours d'Algèbre Supérieure*, t. 2, sec. V, ch. II.

† On this and allied subjects, see Jordan, Traité des substitutions et des équations algébriques, Paris, 1870 (pp. 370-382).

‡ Concerning solvable quintics see McClintock, *American Jour. Math.*, Vol. 6, p. 301.

the equation becomes

$$z^5 - \frac{p^5}{5^5 z^5} + r = 0.$$

We therefore obtain an equation of the second degree for z^5. Hence

$$z = \sqrt[5]{-\frac{r}{2} + \sqrt{Q}}, \qquad Q \equiv \frac{r^2}{4} + \left(\frac{p}{5}\right)^5.$$

Proceeding as in § 172, we obtain the five roots

$$\epsilon^i \sqrt[5]{-\frac{r}{2} + \sqrt{Q}} + \epsilon^{5-i} \sqrt[5]{-\frac{r}{2} - \sqrt{Q}} \qquad (i = 0, 1, 2, 3, 4),$$

where ϵ denotes an imaginary fifth root of unity (Example 1).

The same method is to be followed in working Exercises 9-13 below, the substitution being $y = z - \frac{p}{nz}$ for an equation of degree n, as in (2) and (11).

EXAMPLE 3. The equation $y^5 + py^4 + qy^3 + qy^2 + py + 1 = 0$ may be solved by algebra. It has the root -1. Dividing out the factor $y + 1$, we obtain the quartic equation

$$y^4 + (p-1)y^3 + (q - p + 1)y^2 + (p-1)y + 1 = 0.$$

The latter may be solved by the earlier methods or more simply by setting $y + \frac{1}{y} = z$, whence $y^2 + \frac{1}{y^2} = z^2 - 2$, so that

$$z^2 + (p-1)z + q - p - 1 = 0.$$

$$\therefore \quad z = \frac{1-p}{2} \pm \sqrt{\left(\frac{1+p}{2}\right)^2 + 1 - q}.$$

Then $y = \frac{1}{2}(z + \sqrt{z^2 - 4})$ gives the four values of y.

180. A reciprocal equation is one which is not altered by replacing the variable by its reciprocal. The equations solved in examples 1 and 3 of § 179 are reciprocal equations. Just as these special equations were solved by means of equations of lower degree, so the solution of any reciprocal equation may be made to depend upon the solution of an equation of lower degree.

If the given equation is

$$(12) \quad y^n + p_1 y^{n-1} + p_2 y^{n-2} + \cdots + p_{n-2} y^2 + p_{n-1} y + p_n = 0,$$

the equation obtained by writing $\frac{1}{y}$ for y and multiplying by y^n is

$$p_n y^n + p_{n-1} y^{n-1} + p_{n-2} y^{n-2} + \ldots + p_2 y^2 + p_1 y + 1 = 0.$$

If the two equations are to be the same, we must have

$$p_n p_1 = p_{n-1},\ p_n p_2 = p_{n-2},\ \ldots,\ p_n p_{n-2} = p_2,\ p_n p_{n-1} = p_1,\ p_n^2 = 1.$$

Hence $p_n = \pm 1$, so that there are the two following types:

(I) $p_n = +1,\quad p_1 = p_{n-1}\quad p_2 = p_{n-2},\quad p_3 = p_{n-3}, \ldots,$

(II) $p_n = -1,\quad p_1 = -p_{n-1},\quad p_2 = -p_{n-2},\quad p_3 = -p_{n-3}, \ldots$

Suppose that (12) is a reciprocal equation of type (I). Then

$$(13)\quad (y^n+1)+p_1(y^{n-1}+y)+p_2(y^{n-2}+y^2)+p_3(y^{n-3}+y^3)+\ldots = 0.$$

If n is *odd*, $n = 2t + 1$, the final term in (13) is $p_t(y^{t+1} + y^t)$. The terms are all divisible by $y + 1$ and, if the quotient be $Q(y)$, the equation $Q(y) = 0$, of even degree $n - 1$, is a reciprocal equation. In fact, the dividend (13) and the divisor $y + 1$ remain unaltered, aside from a factor y^m and y, respectively, when $\frac{1}{y}$ is written for y, so that a similar result must hold true for the quotient $Q(y)$. Moreover, $Q(y) = 0$ is of type (I) since the constant term is $+ 1$, being derived from the division of $y^n + 1$ (n odd) by $y + 1$ [see formula (3), Chapter III]. As an illustration see Example 3 above. Hence $Q(y) = 0$ is of the character next treated.

If n is *even*, $n = 2t$, the final term in (13) is $p_t y^t$. Dividing the equation by y^t, we get

$$\left(y^t+\frac{1}{y^t}\right)+p_1\left(y^{t-1}+\frac{1}{y^{t-1}}\right)+p_2\left(y^{t-2}+\frac{1}{y^{t-2}}\right)+\ldots+p_{t-1}\left(y+\frac{1}{y}\right)+p_t=0.$$

Set $y + \frac{1}{y} = z$. Then $y^2 + \frac{1}{y^2} = z^2 - 2,\ \ y^3 + \frac{1}{y^3} = z^3 - 3z$. The general term may be derived from the identity

$$(14)\qquad y^r + \frac{1}{y^r} = z\left(y^{r-1} + \frac{1}{y^{r-1}}\right) - \left(y^{r-2} + \frac{1}{y^{r-2}}\right).$$

$$\therefore \quad y^3 + \frac{1}{y^3} = z(z^2 - 2) - z = z^3 - 3z.$$

$$\therefore \quad y^4 + \frac{1}{y^4} = z(z^3 - 3z) - (z^2 - 2) = z^4 - 4z^2 + 2.$$

$$\therefore \quad y^5 + \frac{1}{y^5} = z(z^4 - 4z^2 + 2) - (z^3 - 3z) = z^5 - 5z^3 + 5z.$$

By mathematical induction (an exercise left to the reader), we get

$$y^m + \frac{1}{y^m} = z^m - mz^{m-2} + \frac{m(m-3)}{1\cdot 2}z^{m-4} - \frac{m(m-4)(m-5)}{1\cdot 2\cdot 3}z^{m-6} + \dots$$

$$+ (-1)^p \frac{m(m-p-1)(m-p-2)\dots(m-2p+2)(m-2p+1)}{1\cdot 2\cdot 3\dots p}z^{m-2p} + \dots$$

Hence equation (13) is reduced to an equation of degree $t = n/2$.

Suppose next that (12) is a reciprocal equation of type (II). If n is odd, $n = 2t + 1$, the equation may be written

$$(y^n - 1) + p_1(y^{n-1} - y) + p_2(y^{n-2} - y^2) + \dots + p_t(y^{t+1} - y^t) = 0,$$

and the left member is divisible by $y - 1$. Calling the quotient $Q(y)$, we note that the dividend and divisor remain unaltered, aside from the respective factors $-y^n$ and $-y$, when $\frac{1}{y}$ is written for y, so that a similar result must hold for $Q(y)$. Hence $Q(y) = 0$ is a reciprocal equation of even degree $n - 1$. Its constant term is $+1$, since it arises from the division of $y^n - 1$ by $y - 1$ [see formula (2), Chapter III]. Hence $Q(y) = 0$ is of type (I), whose reduction to an equation of half the degree was just effected.

If n is even, $n = 2t$, the equation becomes

$$(y^n - 1) + p_1(y^{n-1} - y) + p_2(y^{n-2} - y^2) + \dots + p_{t-1}(y^{t+1} - y^{t-1}) = 0,$$

and the left member is divisible by $y^2 - 1$. Dividing by this factor, we obtain an equation of even degree $n - 2$, which may be shown, as in the previous case, to be a reciprocal equation of type (I).

In addition to the examples 1 and 3 above, consider the equation

$$y^6 - 1 = 0.$$

Removing the factor $y^2 - 1$, we get the reciprocal equation

$$y^4 + y^2 + 1 = 0.$$

Setting $y + \frac{1}{y} = z$, we get $z^2 - 1 = 0$. Hence $y + \frac{1}{y} = \pm 1$, or

$$y^2 - y + 1 = 0, \quad y^2 + y + 1 = 0.$$

EXAMPLE. Solve $y^5 - 5y^4 + 9y^3 - 9y^2 + 5y - 1 = 0$.

Dividing out $y-1$, we get $y^4 - 4y^3 + 5y^2 - 4y + 1 = 0$. Setting $y + \frac{1}{y} = z$, we get $z^2 - 4z + 3 = 0$, whence $z = 1$ or 3. Hence

$$y^2 - y + 1 = 0, \quad \text{or} \quad y^2 - 3y + 1 = 0.$$

$$\therefore \quad y = \tfrac{1}{2}(1 \pm \sqrt{-3}), \quad \tfrac{1}{2}(3 \pm \sqrt{5}), \text{ or } 1.$$

EXERCISES.

Solve the following reciprocal equations:

1. $y^6 + 1 = 0$. 2. $y^7 - 1 = 0$. 3. $y^5 + 1 = 0$.
4. $y^3 - 5y^2 - 5y + 1 = 0$. 5. $y^5 - 7y^4 + y^3 - y^2 + 7y - 1 = 0$.
6. $y^3 - 8y^2 + 8y - 1 = 0$. 7. $y^4 - 10y^3 + 26y^2 - 10y + 1 = 0$.
8. $y^{10} - 3y^8 + 5y^6 - 5y^4 + 3y^2 - 1 = 0$.

Solve, by the method of § 179, Ex. 2, the equations

9. $y^7 + py^5 + \frac{2}{7}p^2y^3 + \frac{1}{7^2}p^3y + r = 0$.
10. $y^9 + py^7 + \frac{3}{9}p^2y^5 + \frac{5}{9^2}p^3y^3 + \frac{1}{9^3}p^4y + r = 0$.
11. $y^{11} + py^9 + \frac{4}{11}p^2y^7 + \frac{7}{11^2}p^2y^5 + \frac{5}{11^3}p^4y^3 + \frac{1}{11^4}p^5y + r = 0$.
12. $y^5 + 10y^3 + 20y + 31 = 0$.
13. $y^5 + 30y^3 + 180y - 211 = 0$.

181. The Fundamental Theorem of Algebra. *An equation of the nth degree has exactly n roots.*

From our solution of the general equations of the degrees 2, 3, and 4, we conclude that they have exactly 2, 3, 4 roots, respectively. In the Appendix it is proved that any equation has a root, whatever be the degree of the equation. It may then be readily shown that an equation of the nth degree has exactly n roots. Let the given equation be

(15) $f(x) \equiv p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_{n-1}x+p_n=0 \quad (p_0 \neq 0).$

By the theorem cited, it has a root, say α, so that $f(\alpha) = 0$. By the factor theorem, $f(x)$ is divisible by $x-\alpha$:

$$f(x) = (x-\alpha)Q(x), \quad Q(x) \equiv p_0x^{n-1}+q_2x^{n-2}+\dots+q_{n-1}.$$

Also the equation $Q(x) = 0$ has a root, say β, so that

$$Q(x) = (x-\beta)Q'(x), \quad Q'(x) \equiv p_0x^{n-2}+rx^{n-3}+\dots$$

$$\therefore \quad f(x) = (x-\alpha)(x-\beta)Q'(x),$$

so that β is also a root of the original equation $f(x) = 0$. Proceeding in this way, we get

(16) $\qquad f(x) \equiv p_0(x-\alpha)(x-\beta)(x-\gamma)\dots(x-\nu).$

But a product vanishes if, and only if, one of its factors vanishes; and $x-\alpha$, for example, vanishes if, and only if, $x = \alpha$. Hence the equation has the n roots $\alpha, \beta, \gamma, \dots, \nu$ and no other roots, if $p_0 \neq 0$.

In case $\alpha = \beta$, with α different from the remaining roots, we say that the equation has a **double root** α. In case $\alpha = \beta = \gamma$, with α different from the remaining roots, the equation is said to have a **triple root**. In general, it may have a **root of multiplicity** 2, 3, 4, . . . , or n.

As shown in §§ 76, 77, an equation of the nth degree having more than n roots has all its coefficients equal to zero and is an equation only in appearance.

If follows that the equation whose roots are $\alpha, \beta, \dots, \nu$ is

$$(x-\alpha)(x-\beta)\dots(x-\nu) = 0.$$

For example, the equation having the roots 1, 2, -1, -2 is

$$(x-1)(x-2)(x+1)(x+2) \equiv (x^2-1)(x^2-4) \equiv x^4-5x^2+4=0.$$

182. Relations between the roots and the coefficients of an equation. As illustrations of the general theorem now to be proved, the case of the quadratic equation (§ 32), the case of the cubic equation (§ 174 and Exercise 7), and the case of the quartic equation (§ 176) should be reviewed by the student.

Suppose that in an equation (15) of the nth degree *the coefficient* p_0 *of the highest power* x^n *is unity.*[*] Comparing it with (16), we have

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n=(x-\alpha)(x-\beta)(x-\gamma)\ldots(x-\nu).$$

Upon multiplying together the n factors in the second member and denoting by S_1 the sum of the roots $\alpha, \beta, \ldots, \mu, \nu$, by S_2 the sum of their products taken two at a time, etc., as written below, we get

$$x^n-S_1x^{n-1}+S_2x^{n-2}-S_3x^{n-3}+\ldots+(-1)^{n-1}S_{n-1}x+(-1)^nS_n.$$

Since this expression must be identical with $x^n+\ldots+p_n$, we get

$$(17)\qquad \begin{cases} -p_1=S_1\equiv\alpha+\beta+\gamma+\ldots+\mu+\nu, \\ \quad p_2=S_2\equiv\alpha\beta+\alpha\gamma+\beta\gamma+\ldots+\gamma\nu+\mu\nu, \\ -p_3=S_3\equiv\alpha\beta\gamma+\alpha\beta\mu+\ldots+\gamma\mu\nu, \\ \quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ (-1)^np_n=S_n\equiv\alpha\beta\gamma\ldots\mu\nu. \end{cases}$$

If the coefficient of x^n *is unity, the negative of the coefficient of* x^{n-1} *equals the sum of the n roots; the coefficient of* x^{n-2} *equals the sum of the products of the roots taken two at a time; the negative of the coefficient of* x^{n-3} *equals the sum of the products of the roots taken three at a time, etc.; finally,* $(-1)^n$ *times the constant term equals the product of the roots.*

Corollary. If the roots of an equation are all positive, the coefficients $1, p_1, p_2, p_3, \ldots p_n$ are alternately positive and negative. If the roots are all negative, the coefficients are all positive.

183. Since we have n relations (17) for the n roots, it might be supposed that these relations would enable us to determine the roots.

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[*] If it is not unity, we divide the equation by p_0 and use the new equation.

To show that the determination of one root α by means of these n relations requires the solution of the given equation of the nth degree, we consider the case $n = 3$, the *method* being general. Then

$$-p_1 = \alpha + \beta + \gamma, \quad p_2 = \alpha\beta + \alpha\gamma + \beta\gamma, \quad -p_3 = \alpha\beta\gamma.$$

Multiply the first relation by α^2, the second by $-\alpha$, and adding to the third relation, we get $-p_1\alpha^2 - p_2\alpha - p_3 = \alpha^3$.

$$\therefore \quad \alpha^3 + p_1\alpha^2 + p_2\alpha + p_3 = 0,$$

which is the given cubic $x^3 + p_1x^2 + p_2x + p_3 = 0$ with α written in the place of x. Evidently any other method of eliminating β and γ from the three relations must lead to the same result.

But the relations between the roots and the coefficients are nevertheless of both theoretical and practical value in problems on the solution of the equation.

184. EXAMPLE 1. Solve the equation $x^3 + 5x^2 - 16x - 80 = 0$, given that one root is the negative of another root.

Since the roots are α, $\beta = -\alpha$, γ, we have

$$-5 = \alpha + \beta + \gamma = \gamma, \quad -16 = \alpha\beta + \alpha\gamma + \beta\gamma = -\alpha^2; \quad 80 = \alpha\beta\gamma = -\alpha^2\gamma.$$

Hence $\gamma = -5$, $\alpha = \pm 4$, $\beta = \mp 4$. The roots are thus -5, 4, -4.

EXAMPLE 2. Solve the equation $x^3 - 15x^2 + 71x - 105 = 0$, being given that the roots are in arithmetical progression.

Denote the roots by $c - d$, c, $c + d$. Their sum $3c$ equals 15, whence $c = 5$. Also

$$(c - d)c + (c - d)(c + d) + c(c + d) \equiv 3c^2 - d^2 = 71.$$

Hence $d^2 = 4$, $d = \pm 2$. The roots are thus 3, 5, 7.

EXAMPLE 3. Solve the equation $x^4 + 20x^3 - 210x^2 - 540x + 729 = 0$, the roots being in geometrical progression.

Denoting the roots by a, ar, ar^2, ar^3, we get

$$\begin{aligned}
-20 &= a(1 + r + r^2 + r^3) = a(1 + r)(1 + r^2),\\
-210 &= a^2(r + r^2 + 2r^3 + r^4 + r^5) = a^2r(1 + r + r^2)(1 + r^2),\\
540 &= a^3(r^3 + r^4 + r^5 + r^6) = a^3r^3(1 + r)(1 + r^2),\\
729 &= a^4r^6.
\end{aligned}$$

By the first and third relations, $a^3r^3 = -27$, and the fourth relation is satisfied. Substituting the value of a^2 in the second relation, it becomes

$$210 = 27\left(1 + r + \frac{1}{r}\right)\left(r + \frac{1}{r}\right),$$

which is evidently a reciprocal equation (§ 180).

$$\therefore \quad r + \frac{1}{r} = \frac{7}{3}, \quad \text{or} \quad \frac{-10}{3}.$$

Using the second value, we get $r = -3$ or $-\frac{1}{3}$. For $r = -3$, we get $a^2 = 1$; but the first relation gives $a = +1$. The roots are 1, -3, 9, -27.

EXAMPLE 4. Solve the equation $6x^3 - 11x^2 + 6x - 1 = 0$, being given that the roots are in harmonical progression.

$$\therefore \quad \frac{1}{c-d} + \frac{1}{c} + \frac{1}{c+d} = \frac{11}{6}, \qquad \frac{1}{c(c-d)} + \frac{1}{c(c+d)} + \frac{1}{c^2 - d^2} = 1,$$

$$\frac{1}{c(c-d)(c+d)} = \frac{1}{6}.$$

By the last two relations, $(c + d) + (c - d) + c = 6$, whence $c = 2$. The third relation then gives $(2 - d)(2 + d) = 3$, whence $d^2 = 1$. The roots are therefore 1, $\frac{1}{2}$, $\frac{1}{3}$.

EXAMPLE 5. Given that one root is double another, solve the equation

$$24x^3 + 14x^2 - 63x - 45 = 0.$$

The roots being α, $\beta = 2\alpha$, and γ, we have

$$3\alpha + \gamma = \frac{-14}{24}, \qquad 2\alpha^2 + 3\alpha\gamma = \frac{-63}{24}, \qquad 2\alpha^2\gamma = \frac{45}{24}.$$

By the first and second relations, $\alpha = \frac{1}{2}$ or $\frac{-3}{4}$, $\gamma = \frac{-25}{12}$ or $\frac{5}{3}$. But the pair of solutions $\alpha = \frac{1}{2}$, $\gamma = \frac{-25}{12}$ do not satisfy the third relation, whereas the pair $\alpha = \frac{-3}{4}$, $\gamma = \frac{5}{3}$ do. Hence the roots are $\frac{-3}{4}$, $\frac{-3}{2}$, $\frac{5}{3}$.

EXERCISES.

Form the equation whose roots are

1. $-3, +3, 1$. 2. $\pm 1, \pm 2, 0$. 3. $c + d, c - d, c, d$.

4. $1, 1 + \sqrt{3}, 1 - \sqrt{3}$. 5. $-1, 2 + 3\sqrt{-1}, 2 - 3\sqrt{-1}$.

6. Solve $4x^3 - 16x^2 - 9x + 36 = 0$, one root being the negative of another.

7. Solve $x^3 - 10x^2 + 27x - 18 = 0$, one root being double another.

8. Solve $x^3 - 9x^2 + 23x - 15 = 0$, one root being triple another.

9. Solve $x^4 - 4x^3 - 2x^2 + 12x + 9 = 0$, having two pairs of equal roots.

10. Solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, whose roots are in arithmetical progression, say $c - 3b$, $c - b$, $c + b$, $c + 3b$.

11. Solve $x^3 - 14x^2 - 84x + 216 = 0$, whose roots are in A.P.

12. Solve $27x^3 - 42x^2 - 28x + 8 = 0$, whose roots are in G.P.

13. Solve $3x^4 - 40x^3 + 130x^2 - 120x + 27 = 0$, whose roots are in G.P.

14. Solve $24x^3 - 26x^2 + 9x - 1 = 0$, whose roots are in H.P.

15. Solve $81x^3 - 18x^2 - 36x + 8 = 0$, whose roots are in H.P.

16. Solve the equation in Ex. 15, one root being the negative of another.

17. Solve the equation in Ex. 14, one root being double another.

18. If the roots of $x^3 - px^2 + qx - r = 0$ are in G.P., then $q^3 = p^3r$.

19. If one root of $x^3 - px^2 + qx - r = 0$ is the negative of another, $pq = r$.

20 Solve $x^4 - 4x^3 + 12x^2 - 10x + 3 = 0$, having a triple root.

21. Solve $x^5 - 13x^4 + 67x^3 - 171x^2 + 216x - 108 = 0$, having a double root and a triple root.

185. Fractional Roots. *An equation whose coefficients are* **integers**, *that of the first term being unity, has no fractional root.*

For if the fraction $\frac{a}{b}$, where a and b are integers having no common divisor greater than unity, is a root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n = 0,$$

then, setting $x = \frac{a}{b}$ and multiplying the equation by b^{n-1},

$$\frac{a^n}{b} + p_1a^{n-1} + p_2a^{n-2}b + \dots + p_{n-1}ab^{n-2} + p_nb^{n-1} = 0,$$

so that the fraction $\frac{a^n}{b}$ would be expressible as an integer, which is contrary to the hypothesis on a and b.

186. Surd Roots. *If an equation with* **rational** *coefficients has the root $a + \sqrt{b}$, where a and b are rational, but $\sqrt{b}$ is irrational, then the equation has the root $a - \sqrt{b}$.*

Let $a+\sqrt{b}$ be a root of $f(x)=0$, an equation with rational coefficients. Consider the product

$$(x-a-\sqrt{b})(x-a+\sqrt{b}) \equiv (x-a)^2-b \equiv x^2-2ax+a^2-b.$$

Divide this quadratic expression whose coefficients are rational into the function $f(x)$ and denote the quotient by Q and the remainder by $Rx+R'$, where R and R' are rational numbers independent of x.

$$\therefore\quad f(x) \equiv Q\{(x-a)^2-b\}+Rx+R'.$$

This identity holds for any value of x. Taking $x=a+\sqrt{b}$, the left member is zero by hypothesis, also $(x-a)^2-b$ vanishes.

$$\therefore\quad 0=R(a+\sqrt{b})+R'.$$

By § 5, this is possible if, and only if,

$$0=Ra+R',\quad 0=R.$$

Hence $R=0$, $R'=0$, so that $f(x)$ is divisible by $(x-a)^2+b$ and therefore by the factor $x-a+\sqrt{b}$ of the latter. Hence $a-\sqrt{b}$ is a root of $f(x)=0$.

187. Imaginary roots. *If an equation with* **real** *coefficients has the root $a+b\sqrt{-1}$, where a and b are real, it has also the root $a-b\sqrt{-1}$, so that imaginary roots occur in pairs.*

Let $a+ib$, where $i=\sqrt{-1}$, be a root of $f(x)=0$, an equation with real coefficients. Consider the product

$$(x-a-ib)(x-a+ib) \equiv (x-a)^2+b^2.$$

Divide $f(x)$ by $(x-a)^2+b^2$ and let the quotient be Q and the remainder be $Rx+R'$, where R and R' are real.

$$\therefore\quad f(x) \equiv Q\{(x-a)^2+b^2\}+Rx+R'.$$

For $x = a + ib$, $f(x) = 0$ by hypothesis. Hence

$$0 = R(a + ib) + R'.$$

By § 9, this is possible if, and only if,

$$0 = Ra + R', \quad Rb = 0.$$

Hence $R = 0$, $R' = 0$, so that $f(x)$ is divisible by $(x-a)^2 + b^2$ and therefore by the factor $x-a+ib$. Hence $a-ib$ is a root of $f(x)=0$.

EXERCISES.

1. Solve $x^3 - 10x^2 + 27x - 18 = 0$, the roots being rational.
2. Solve $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, the roots being rational.
3. Solve $x^4 - 4x^3 + 4x - 1 = 0$, one root being $2 + \sqrt{3}$.
4. Solve $x^4 - 36x^2 - 72x - 36 = 0$, one root being $-3 + \sqrt{3}$.
5. Solve $x^4 - 4x^3 + 5x^2 - 2x - 2 = 0$, one root being $1 - \sqrt{-1}$.
6. Solve $x^3 - 3x^2 - 6x - 20 = 0$, one root being $-1 + \sqrt{-3}$.
7. Solve $x^3 - (4 + \sqrt{3})x^2 + (5 + 4\sqrt{3})x - 5\sqrt{3} = 0$, one root being $2 + \sqrt{-1}$.
8. If $\sqrt{3}$ is a root of the equation in Ex. 7, is $-\sqrt{3}$ also a root?
9. A root of $x^3-(i + 5)x^2 + (4i + 9)x-5i-5=0$ is $1+i$. Is $1 - i$ a root?
10. Solve $x^5 + x^4 - 8x^2 - 9x + 15 = 0$, with the roots $\sqrt{3}$ and $-1-2\sqrt{-1}$.

188. Symmetric functions of the roots. A function of the roots $\alpha, \beta, \gamma, \ldots, \mu, \nu$ of an equation

$$x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0 \tag{18}$$

is said to be symmetrical (§ 33) if all the roots are similarly involved. For example, the functions

$$\alpha + \beta + \gamma + \ldots + \mu + \nu, \quad \alpha\beta + \alpha\gamma + \beta\gamma + \ldots + \mu\nu,$$
$$\alpha\beta\gamma + \ldots + \gamma\mu\nu, \ldots, \alpha\beta\gamma \ldots \mu\nu$$

are symmetric functions of the roots, and their values are $-p_1, p_2, -p_3, \ldots, (-1)^n p_n$, respectively (§ 182). More generally, any rational symmetric function of the roots can be expressed rationally in terms of the coefficients of the equation. We shall prove certain interesting cases of this theorem.

EXAMPLE 1. To find the sum of the squares of the roots.

$$\alpha^2+\beta^2+\gamma^2+\dots+\mu^2+\nu^2=(\alpha+\beta+\gamma+\dots+\mu+\nu)^2-2(\alpha\beta+\alpha\gamma+\dots+\mu\nu)$$
$$=(-p_1)^2-2(p_2)=p_1^2-2p_2.$$

EXAMPLE 2. To find $\alpha^2\beta+\alpha\beta^2+\alpha^2\gamma+\alpha\gamma^2+\dots+\mu^2\nu+\mu\nu^2\equiv\Sigma\alpha^2\beta$. Multiplying together the equations

$$\alpha+\beta+\gamma+\dots+\mu+\nu=-p_1, \quad \alpha\beta+\alpha\gamma+\beta\gamma+\dots+\mu\nu=p_2,$$

we get $$\Sigma\alpha^2\beta+3\alpha\beta\gamma+3\alpha\beta\mu+\dots+3\gamma\mu\nu=-p_1p_2.$$
$$\therefore\ \Sigma\alpha^2\beta=3p_3-p_1p_2.$$

EXAMPLE 3. To find the sum of the cubes of the roots.

$$(\alpha+\beta+\dots+\mu+\nu)(\alpha^2+\beta^2+\dots+\mu^2+\nu^2)$$
$$=(\alpha^3+\beta^3+\dots+\nu^3)+(\alpha^2\beta+\alpha\beta^2+\dots+\mu^2\nu+\mu\nu^2)$$

Applying the results of Examples 1 and 2, we get

$$\alpha^3+\beta^3+\dots+\nu^3=(-p_1)(p_1^2-2p_2)-(3p_3-p_1p_2)=-p_1^3+3p_1p_2-3p_3$$

189. Denote by $\Sigma_1, \Sigma_2, \Sigma_3, \dots, \Sigma_r$ the sum of the 1st, 2nd, 3rd, . . . , rth powers, respectively, of the roots. Then $\Sigma_1=-p_1$, $\Sigma_2=p_1^2-2p_2$, $\Sigma_3=-p_1^3+3p_1p_2-3p_3$, by Examples 1 and 3 of the last section. These results give the relations

$$(19)\quad \Sigma_1+p_1=0, \quad \Sigma_2+p_1\Sigma_1+2p_2=0, \quad \Sigma_3+p_1\Sigma_2+p_2\Sigma_1+3p_2=0.$$

Another relation of similar form is derived by substituting in turn $\alpha, \beta, \gamma, \dots, \nu$ for x in the given equation (18). Then

$$\alpha^n+p_1\alpha^{n-1}+p_2\alpha^{n-2}+\dots+p_{n-1}\alpha+p_n=0,$$
$$\beta^n+p_1\beta^{n-1}+p_2\beta^{n-2}+\dots+p_{n-1}\beta+p_n=0, \dots$$

[illegible]

$$\ldots \Sigma_1 - np_n = 0.$$

[illegible] $\overset{=}{<} n$,

$$\ldots \Sigma_1 - rp_r = 0 \quad \left(r \overset{=}{<} n\right).$$

[illegible] terms of the [illegible]

$$\ldots \alpha^r + \beta^r + \ldots + \nu^r),$$

$$\ldots - \mu\nu^{r-1} + \mu^{r-1}\nu),$$

$$\ldots - \mu^{r-2}\nu)$$

$$\ldots - \alpha\gamma\beta^{r-2} + \ldots),$$

$$\ldots - \alpha\beta\gamma\nu^{r-3} + \ldots),$$

. . . .

$$\ldots \alpha\beta\gamma \ldots \rho\sigma + \ldots),$$

[illegible] we observe that each quantity in [illegible] the quantity in the [illegible] so that the sum is zero.

[illegible] except perhaps that for $p_{r-1}\Sigma_1$ which [illegible] $\ldots \pi\sigma - \alpha\beta\gamma \ldots \rho\sigma + \ldots)$ [illegible] $\ldots - r$. There is a single [illegible] in the product, but r ways to reach the [illegible] $\gamma, \ldots, \pi, \rho$, or σ from the [illegible]

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Adding the resulting n relations, we get

$$(20)\qquad \Sigma_n + p_1\Sigma_{n-1} + p_2\Sigma_{n-2} + \ldots + p_{n-1}\Sigma_1 + np_n = 0.$$

We proceed to show that, if r is any positive integer $\overline{\overline{<}}\, n$,

$$(21)\quad \Sigma_r + p_1\Sigma_{r-1} + p_2\Sigma_{r-2} + \ldots + p_{r-1}\Sigma_1 + rp_r = 0 \quad \left(r \overline{\overline{<}}\, n\right).$$

Substitute for $p_1, \ldots, p_r$ their expressions (17) in terms of the roots. Then*

$$\begin{aligned}
\Sigma_r &= && (\alpha^r + \beta^r + \ldots + \nu^r),\\
p_1\Sigma_{r-1} &= -(\alpha^r + \beta^r + \ldots + \nu^r)\\
&\qquad - (\alpha\beta^{r-1} + \alpha^{r-1}\beta + \ldots + \mu\nu^{r-1} + \mu^{r-1}\nu),\\
p_2\Sigma_{r-2} &= \quad (\alpha\beta^{r-1} + \alpha^{r-1}\beta + \ldots + \mu\nu^{r-1} + \mu^{r-1}\nu)\\
&\qquad + (\alpha\beta\gamma^{r-2} + \alpha\gamma\beta^{r-2} + \ldots),\\
p_3\Sigma_{r-3} &= -(\alpha\beta\gamma^{r-2} + \alpha\gamma\beta^{r-2} + \ldots)\\
&\qquad - (\alpha\beta\gamma\delta^{r-3} + \alpha\beta\gamma\nu^{r-3} + \ldots),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
p_{r-1}\Sigma_1 &= (-1)^{r-1}(\alpha\beta\gamma\ldots\rho^2 + \alpha\beta\ldots\rho\sigma^2 + \ldots)\\
&\qquad + (-1)^{r-1}r(\alpha\beta\gamma\ldots\rho\sigma + \ldots),\\
rp_r &= (-1)^r r(\alpha\beta\gamma\ldots\rho\sigma + \ldots).
\end{aligned}$$

Upon adding the second members, we observe that each quantity in a parenthesis on the extreme right cancels the quantity in the parenthesis on the left in the line below, so that the sum is zero.

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* The relations follow immediately, except perhaps that for $p_{r-1}\Sigma_1$ which equals the product of $(-1)^{r-1}(\alpha\beta\gamma\ldots\pi\rho + \alpha\beta\gamma\ldots\pi\sigma + \alpha\beta\gamma\ldots\rho\sigma + \ldots)$ and $\Sigma_1 \equiv \alpha + \beta + \gamma + \ldots + \pi + \rho + \sigma + \ldots + \nu$. There is a single way to reach the term $\alpha\beta\gamma\ldots\pi\rho^2$ in the product, but r ways to reach the term $\alpha\beta\gamma\ldots\pi\rho\sigma$, viz., by using either $\alpha, \beta, \gamma, \ldots, \pi, \rho$, or σ from the sum Σ_1.

Suppose next that $r > n$. For the case $r = n + 1$, we retain the above relations with the exception of the last two. The one preceding the last is no longer true, the correct result being now

$$\begin{aligned} p_n \Sigma_1 &= (-1)^n(\alpha\beta\gamma\ldots\mu\nu)(\alpha + \beta + \gamma + \ldots + \mu + \nu) \\ &= (-1)^n(\alpha\beta\gamma \ldots \mu\nu^2 + \alpha\beta\gamma \ldots \mu^2\nu + \ldots + \alpha^2\beta\gamma \ldots \mu\nu). \end{aligned}$$

Upon adding the latter to the n preceding relations, we get

$$\Sigma_{n+1} + p_1 \Sigma_n + p_2 \Sigma_{n-1} + \cdots + p_{n-1} \Sigma_2 + p_n \Sigma_1 = 0.$$

In a similar manner, we may prove the formula

$$(22)\ \Sigma_{n+m} + p_1 \Sigma_{n+m-1} + p_2 \Sigma_{n+m-2} + \cdots + p_{n-1} \Sigma_{m+1} + p_n \Sigma_m = 0.$$

But the last formula is derived more simply as follows. Multiplying the given equation (18) by x^m, we get

$$x^{n+m} + p_1 x^{n+m-1} + p_2 x^{n+m-2} + \ldots + p_{n-1} x^{m+1} + p_n x^m = 0.$$

Setting $x = \alpha, \beta, \gamma, \ldots, \mu, \nu$ in turn and adding the n resulting relations, we evidently obtain formula (22).

190. Relations (21), which include relations (9), enable us to express $\Sigma_1, \Sigma_2, \Sigma_3, \ldots, \Sigma_n$ in terms of the coefficients. We have only to take $r = 1, 2, 3, \ldots, n$ in turn, the respective equations giving $\Sigma_1, \Sigma_2, \Sigma_3, \ldots, \Sigma_n$ in turn. For example,

$$\begin{aligned} \Sigma_2 &= -p_1 \Sigma_1 - 2p_2 = p_1^2 - 2p_2, \\ \Sigma_3 &= -p_1 \Sigma_2 + p_1 p_2 - 3p_3 = -p_1^3 + 3p_1p_2 - 3p_3, \\ \Sigma_4 &= -p_1 \Sigma_3 - p_2 \Sigma_2 - p_3 \Sigma_1 - 4p_4 \\ &= p_1^4 - 4p_1^2p_2 + 4p_1 p_3 + 2p_2^2 - 4p_4. \end{aligned}$$

Then by employing formula (22) for $m = 1, 2, 3, \ldots$ in turn, we may express $\Sigma_{n+1}, \Sigma_{n+2}, \Sigma_{n+3}, \ldots$ in terms of the coefficients.

EXERCISES

1. If α, β, γ are the roots of $x^3 + p_1 x^2 + p_2 x + p_3 = 0$, find

$$\Sigma_3,\ \Sigma_4,\ \Sigma_5,\ \Sigma_6,\quad (\alpha^2\beta^2 + \alpha^2\gamma^2 + \beta^2\gamma^2),\quad (\alpha + \beta)(\alpha + \gamma)(\beta + \gamma).$$

2. If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + p_1 x^3 + p_2 x^2 + p_3 x + p_4 = 0$, find

$$\Sigma_3,\ \Sigma_4,\ \Sigma_5,\ \Sigma_6,\quad (\alpha^2\beta\gamma + \alpha^2\beta\delta + \ldots + \beta\gamma\delta^2),\quad (\alpha^3\beta + \alpha\beta^3 + \ldots).$$

3. Find the sum of the reciprocals of the roots of the equations in Examples 1 and 2. Find the sum of the squares of their reciprocals.

4. Show that for the general equation (18)

$$(\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 + \ldots = n\,\Sigma_2 - \Sigma_1^2 \equiv \begin{vmatrix} n & \Sigma_1 \\ \Sigma_1 & \Sigma_2 \end{vmatrix},$$

$$(\alpha - \beta)^2(\alpha - \gamma)^2(\beta - \gamma)^2 + \ldots = \begin{vmatrix} n & \Sigma_1 & \Sigma_2 \\ \Sigma_1 & \Sigma_2 & \Sigma_3 \\ \Sigma_2 & \Sigma_3 & \Sigma_4 \end{vmatrix}.$$

5. Solving equations (19) and (21) by determinants, prove that

$$\Sigma_2 = \begin{vmatrix} p_1 & 1 \\ 2p_2 & p_1 \end{vmatrix},\quad \Sigma_3 = -\begin{vmatrix} p_1 & 1 & 0 \\ 2p_2 & p_1 & 1 \\ 3p_3 & p_2 & p_1 \end{vmatrix},\quad \Sigma_4 = \begin{vmatrix} p_1 & 1 & 0 & 0 \\ 2p_2 & p_1 & 1 & 0 \\ 3p_3 & p_2 & p_1 & 1 \\ 4p_4 & p_3 & p_2 & p_1 \end{vmatrix}, \ldots$$

$$2p_2 = \begin{vmatrix} \Sigma_1 & 1 \\ \Sigma_2 & \Sigma_1 \end{vmatrix},\quad 6p_3 = -\begin{vmatrix} \Sigma_1 & 1 & 0 \\ \Sigma_2 & \Sigma_1 & 2 \\ \Sigma_3 & \Sigma_2 & \Sigma_1 \end{vmatrix},\quad 24p_4 = \begin{vmatrix} \Sigma_1 & 1 & 0 & 0 \\ \Sigma_2 & \Sigma_1 & 2 & 0 \\ \Sigma_3 & \Sigma_2 & \Sigma_1 & 3 \\ \Sigma_4 & \Sigma_3 & \Sigma_2 & \Sigma_1 \end{vmatrix}, \ldots$$

6. Form the equations whose roots are (1) the squares, (2) the cubes, of the roots of $x^3 + 3x^2 + x - 7 = 0$.

7. If α, β, γ are the roots of $x^3 + p_1 x^2 + p_2 x + p_3 = 0$, form the equations whose roots are (1) $\alpha^3, \beta^3, \gamma^3$; (2) $\alpha^{-2}, \beta^{-2}, \gamma^{-2}$; (3) $\alpha\beta, \alpha\gamma, \beta\gamma$; (4) $\frac{2}{\alpha}, \frac{2}{\beta}, \frac{2}{\gamma}$; (5) $\frac{\alpha + \beta}{\gamma}, \frac{\alpha + \gamma}{\beta}, \frac{\beta + \gamma}{\alpha}$; (6) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}, \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha}, \frac{\beta}{\gamma} + \frac{\gamma}{\beta}$.

8. Form the equation of lowest degree with rational coefficients, one of whose roots is (1) $\sqrt{2} + \sqrt{5}$; (2) $\sqrt{2 + \sqrt{3}}$; (3) $\sqrt{3} - \sqrt{-1}$; (4) $\sqrt{-4} + \sqrt{-3}$.

MISCELLANEOUS EXERCISES.

1. $1^4 + 2^4 + \dots + n^4 = \frac{1}{30} n(n+1)(2n+1)(3n^2+3n-1)$.

2. $1^5 + 2^5 + \dots + n^5 = \frac{1}{12} n^2(n+1)^2(2n^2+2n-1)$.

3. $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} +$ *ad infinitum* $= \frac{1}{2}$.

4. The arithmetical mean of two positive quantities is greater than their geometrical mean.

5. According as $a = b$ or $a \neq b$, $a^2 + b^2 = 2ab$ or $> 2ab$.

6. If $(a+b+c)^2 = 3(a^2+b^2+c^2)$, then $a = b = c$.

7. If $(a_1+a_2+\dots+a_n)_1^2 = n(a_2^2+a_1^2+\dots+a_n^2)$, then $a_1 = a_2 = \dots = a_n$.

8. According as $x+y+z >, =,$ or < 0, $x^3+y^3+z^2 >, =,$ or $< 3xyz$.

9. If x, y, z are real and not all equal, then

$$27xyz < (x+y+z)^3 < 9(x^3+y^3+z^3).$$

10. Prove by mathematical induction that

$$a + (a+d) + (a+2d) + \dots + (a + \overline{n-1}d) = n\left(a + \frac{n-1}{2}d\right).$$

11. Show that the roots of $x^3 - 7x + 7 = 0$ are

$$1.3568958\dots,\quad 1.69\dots,\quad -3.048\dots$$

12. The roots of $x^3 + x^2 - 2x - 1 = 0$ are 1.24698, -1.80194, $-.44504$.

13. The positive root of $7x^4 + 20x^3 + 3x^2 - 16x - 8 = 0$ is 0.91336.

14. $(\alpha-\beta)^2 + (\alpha-\gamma)^2 + (\beta-\gamma)^2 = 3(\alpha^2+\beta^2+\gamma^2) - (\alpha+\beta+\gamma)^2$.

15. $$\begin{vmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{vmatrix} \equiv (af - be + cd)^2.$$

16. $$\begin{vmatrix} \beta+\gamma & \gamma+\alpha & \alpha+\beta \\ \beta_1+\gamma_1 & \gamma_1+\alpha_1 & \alpha_1+\beta_1 \\ \beta_2+\gamma_2 & \gamma_2+\alpha_2 & \alpha_2+\beta_2 \end{vmatrix} \equiv 2\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}.$$

17. Factor $x^4 - 17x^2 + 20x - 6$, $x^4 + 8x^3 + 21x^2 + 26x + 14$.

18. $x^4 + x^3 - 2x^2 + 4x - 24 = 0$ has two real and two imaginary roots.

19. Factor $(a+1)^7 - a^7 - 1$.

20. Solve the reciprocal equation $27x^2(1-x)^2 = 4(1-x+x^2)^3$.

21. If α, β, γ are the roots of $x^3 + x^2 - 2x - 1 = 0$, form the equation whose roots are $\alpha^2+\beta^2$, $\alpha^2+\gamma^2$, $\beta^2+\gamma^2$.

22. Sum to n terms $\frac{1}{3!} + \frac{5}{4!} + \frac{11}{5!} + \frac{19}{6!} + \dots$

23. Sum to infinity $\frac{1\cdot 2}{3} + \frac{2\cdot 3}{3^2} + \frac{3\cdot 4}{3^3} + \dots$

24. $\frac{2}{1!}+\frac{12}{2!}+\frac{28}{3!}+\frac{50}{4!}+\frac{78}{5!}+\frac{112}{6!}+\ldots=5e+2.$

25. $1^3+\frac{2^3}{1!}+\frac{3^3}{2!}+\frac{4^3}{3!}+\ldots=15e.$

26. Find the number of shot arranged in a triangular pyramid, there being 25 shot along each edge.

27. If the sides of a right-angled triangle are in A. P., they are proportional to 3, 4, 5.

28. At tennis A and B play together with the respective winning probabilities a and b. If the score is deuce, show that A's probability of winning the game is

$$a^2+2a^3b+4a^4b^2+8a^5b^3+\ldots=\frac{a^2}{1-2ab}.$$

APPENDIX.

Argand's Diagram. In 1806 Argand gave a method of representing complex numbers by points in a plane. The complex number $x + y\sqrt{-1}$ is represented by the point (x, y) whose coördinates referred to two fixed perpendicular lines Ox and Oy are the real numbers x and y. To each complex number $x + iy$ there corresponds a single point P, and to every point of the plane there corresponds one and but one complex number. If the point P moves in the plane, $z \equiv x + iy$ changes continually in value and is called a **complex variable.** The length of the line OP, viz.,

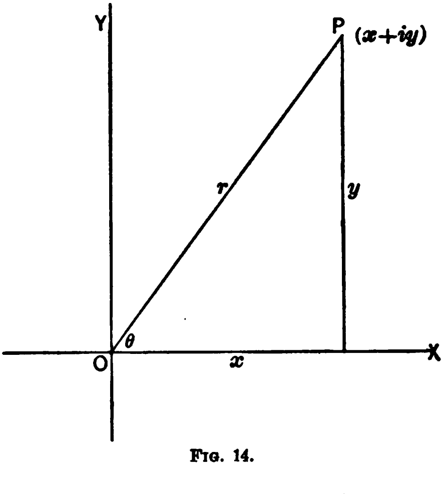

Fig. 14.

$\sqrt{x^2+y^2}$, is called the **modulus** of $x+iy$. The angle $\theta \equiv XOP$ is called the **argument** of $x+iy$; it is to be measured counter-clockwise from OX.

By trigonometry, the **sine** of θ is the ratio of y to r, the **cosine** of θ is the ratio of x to r. These relations are written

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \text{whence } \sin^2\theta + \cos^2\theta = 1.$$

$$\therefore \quad x + iy = r(\cos\theta + i\sin\theta).$$

For example, $\sqrt{2} + i\sqrt{2} = 2(\cos 45^\circ + i\sin 45^\circ)$,

$-\frac{1}{2}+\frac{1}{2}\sqrt{-3} = \cos 120^\circ + i\sin 120^\circ$, $\frac{1}{2}+\frac{1}{2}\sqrt{3}\,i = \cos 60^\circ + i\sin 60^\circ$.

If the points P and A represent $x+iy$ and $a+i\alpha$, respectively, it is seen that the point which represents their sum $(x+a)+i(y+\alpha)$ is given by completing the parallelogram two of whose sides are OP and OA. It follows that the modulus of the sum (the length of the diagonal) of two complex numbers is less than (in a special case, equal to) the sum of their moduli.

The point representing the product of two complex numbers may be constructed by applying the following theorem:

The modulus of the product of two complex numbers is the product of their moduli, its argument is the sum of their arguments.

Let
$$x + iy = r(\cos\theta + i\sin\theta),$$
$$x' + iy' = r'(\cos\theta' + i\sin\theta').$$

Then
$$(x+iy)(x'+iy') = rr'(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta')$$
$$= rr'\{(\cos\theta\cos\theta' - \sin\theta\sin\theta') + i(\sin\theta\cos\theta' + \cos\theta\sin\theta')\}$$
$$= rr'\{\cos(\theta+\theta') + i\sin(\theta+\theta')\},$$

by the formulæ proved in trigonometry.

Demoivre's Theorem. *If n be any rational number and θ any angle,* $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

We have just shown that

$$(\cos\theta + i\sin\theta)(\cos\theta' + i\sin\theta') = \cos(\theta+\theta') + i\sin(\theta+\theta').$$

Taking first $\theta' = \theta$, we have

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta.$$

Taking next $\theta' = 2\theta$, and applying the last result, we get

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

Taking next $\theta' = 3\theta$, and applying the previous result, we get

$$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta.$$

Similarly, if the theorem be true for $n = r$, a positive integer, it is true for $n = r + 1$. Hence, by mathematical induction, it is true for *every* positive integral value of n.

Replacing θ by $\frac{\theta}{n}$, we get

$$\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)^n = \cos\theta + i\sin\theta.$$

$$\therefore (\cos\theta + i\sin\theta)^{\frac{1}{n}} = \cos\frac{\theta}{n} + i\sin\frac{\theta}{n},$$

as one of the nth roots of $\cos\theta + i\sin\theta$. It follows that the theorem is true for positive fractional values of n.

Next, if $n = -m$, m being a positive fraction,

$$(\cos\theta + i\sin\theta)^n = \frac{1}{(\cos\theta + i\sin\theta)^m}$$

$$= \frac{1}{\cos m\theta + i\sin m\theta} = \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}$$

$$= \cos m\theta - i\sin m\theta = \cos n\theta + i\sin n\theta.$$

Corollary. Any root of a complex number may be expressed as a complex number. Thus $\sqrt[n]{x + iy} = \sqrt[n]{r}\,(\cos\theta + i\sin\theta)^{\frac{1}{n}}$. Hence $\sqrt[n]{r}\,\cos\left(\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ is one of the nth roots of $x + iy$. Replacing θ by $\theta + 360°$, $\theta + 2\cdot 360°, \ldots, \theta + (n-1)\,360°$, we obtain the remaining $n - 1$ roots. But $\theta + n\,360°$ gives the same root as θ, $\theta + (n+1)\,360°$ gives the same root as $\theta + 360°$, etc.

Example 1. Find the three cube roots of unity.

$$1=\cos 0+i\sin 0=\cos 360^\circ+i\sin 360^\circ=\cos 720^\circ+i\sin 720^\circ.$$

Hence the cube roots of unity are

$$\cos\tfrac{0}{3}+i\sin\tfrac{0}{3}=1,\quad \cos 120^\circ+i\sin 120^\circ=-\tfrac{1}{2}+\tfrac{1}{2}\sqrt{-3}\equiv\omega,$$
$$\cos 240^\circ+i\sin 240^\circ=-\tfrac{1}{2}-\tfrac{1}{2}\sqrt{-3}=\omega^2.$$

Example 2. Find the five fifth roots of -1.

$$-1=\cos 180^\circ+i\sin 180^\circ=\cos 540^\circ+i\sin 540^\circ=\ldots=\cos 1620^\circ+i\sin 1620^\circ.$$
$$\therefore\ \sqrt[5]{-1}=\cos\ 36^\circ+i\sin 36^\circ,\text{ or }\cos 108^\circ+i\sin 108^\circ,\ \ldots,\text{ or }\cos 324^\circ+i\sin 324^\circ.$$

Solution of the cubic $y^3+py+r=0$ *in the irreducible case* (*see* § 173):

$$R\equiv\frac{r^2}{4}+\frac{p^3}{27}<0. \tag{1}$$

Set $\quad R\equiv-\rho^2\sin^2\theta,\quad -\dfrac{r}{2}=\rho\cos\theta.$

$$\therefore\ \rho^2\equiv\rho^2(\cos^2\theta+\sin^2\theta)=\frac{r^2}{4}-R=-\frac{p^3}{27}.$$
$$\therefore\ \rho=\sqrt{-p^3/27},\quad \cos\theta=-\frac{r}{2}\div\sqrt{-p^3/27}.$$

Notice that p must be negative by (1), so that ρ is *real;* also that $-\dfrac{r}{2}$ is in numerical value less than $\sqrt{-p^3/27}$ by (1), so that $\cos\theta$ is numerically less than unity. Hence θ may be found by means of a trigonometric table of cosines.

$$\therefore\ \sqrt[3]{-\frac{r}{2}+\sqrt{R}}=\sqrt[3]{\rho\cos\theta+i\rho\sin\theta}$$
$$=\sqrt[3]{\rho}\left\{\cos\frac{\theta+m\cdot 360^\circ}{3}+i\sin\frac{\theta+m\cdot 360^\circ}{3}\right\},$$
$$\sqrt[3]{-\frac{r}{2}-\sqrt{R}}=\sqrt[3]{\rho}\left\{\cos\frac{\theta+m\cdot 360^\circ}{3}-i\sin\frac{\theta+m\cdot 360^\circ}{3}\right\},$$

where $m=0$, 1, or 2 and $\sqrt[3]{\rho}$ is a real quantity. Substituting in Cardan's formulæ, we obtain the three roots

$$y=2\sqrt[3]{\rho}\cos\frac{\theta+m\cdot 360^\circ}{3}\quad(m=0,\ 1,\text{ and }2).$$

We may proceed independently of Cardan's formulæ. By trigonometry, $\cos 3x = 4\cos^3 x - 3\cos x$. Set $z = \cos x$.

$$\therefore z^3 - \tfrac{3}{4}z - \tfrac{1}{4}\cos 3x = 0.$$

To reduce the given cubic to this form, set $y = kz$.

$$\therefore z^3 + \frac{p}{k^2}z + \frac{r}{k^3} = 0.$$

Determine k so that $\dfrac{p}{k^2} = \dfrac{-3}{4}$, viz., take $k = \sqrt{\dfrac{-4p}{3}}$. The two equations will then be identical if

$$-\frac{1}{4}\cos 3x = \frac{r}{k^3} = \frac{r}{8} \div \sqrt{\frac{-p^3}{27}}.$$

Hence $\cos 3x = -\dfrac{r}{2} \div \sqrt{-p^3/27}$, the value found above for $\cos\theta$ Finding $3x$ from its cosine by means of a table, a root z equals $\cos x$. But $\cos 3x = \cos(3x + 360°) = \cos(3x + 720°)$. Hence the three roots z are

$$\cos x, \quad \cos(x + 120°), \quad \cos(x + 240°).$$

Multiplying these by $k = 2\sqrt{-p/3}$, we get the three roots y.

FUNDAMENTAL THEOREM.

Every equation $z^n + p_1 z^{n-1} + \ldots + p_n = 0$ *has a root.*

If we set $w \equiv z^n + p_1 z^{n-1} + p_2 z^{n-2} + \ldots + p_n$, we are to prove that there exists at least one complex number z which will make $w = 0$, so that the point (called the w-point) in Argand's diagram which represents w will be the origin O. Unless the theorem is true, there is a w_1-point, say P_1, corresponding to some value z_1 of z and distinct from O, such that no other w-point is nearer to O than P_1, so that there is no value of z which will bring the corresponding w-point within the circle about O of radius OP_1. If we prove that the last statement is false, the theorem will be proved.

Let us change z from z_1 to $z_1 + Z_1$ by adding a small quantity Z_1. Then w will change from w_1 to W, where

$$W=(z_1+Z_1)^n+p_1(z_1+Z_1)^{n-1}+p_2(z_1+Z_1)^{n-2}+\ldots+p_{n-1}(z_1+Z_1)+p_n$$
$$=z_1+p_1z_1^{n-1}+p_2z_1^{n-2}+\ldots+p_{n-1}z_1+p_n+AZ_1+BZ_1^2+\ldots+Z_1^n,$$

upon expanding the powers by the binomial theorem and arranging the terms according to powers of Z_1. The coefficient of Z_1 is seen to be*

$$A \equiv nz_1^{n-1} + (n-1)p_1z_1^{n-2} + (n-2)p_2z_1^{n-3} + \ldots + p_{n-1}.$$

Since $w_1 \equiv z_1 + p_1z_1^{n-1} + \ldots + p_n$, we get

$$W = w_1 + AZ_1 + BZ_1^2 + \ldots + Z_1^n.$$

Set $A \equiv a\,(\cos\alpha + i\sin\alpha)$, $Z_1 = r\,(\cos\theta + i\sin\theta)$. As above,

$$AZ_1 = ar\,[\cos(\theta+\alpha) + i\sin(\theta+\alpha)].$$

$$\therefore\ W = w_1+ar\,[\cos(\theta+\alpha)+i\sin(\theta+\alpha)]+C_2r^2+C_3r^3+\ldots+C_nr^n,$$

the exact form of $C_2 \equiv B\,(\cos\theta + i\sin\theta)^2$, $C_3, \ldots, C_n$ being immaterial. By taking r sufficiently small, the value of $C_2r^2 + \ldots + C_nr^n$ can be made as small as we please, and therefore W will differ as little as we please from

$$w_1 + ar\,[\cos(\theta+\alpha) + i\sin(\theta+\alpha)].$$

But the point representing this complex quantity will describe a circle of radius ar about the point P_1 when the angle θ increases gradually from 0° to 360°. But this circle intersects the circle about O with radius OP_1. Hence by assigning suitable values to r and θ, the w-point may be brought into coincidence with some W-point lying inside the circle of radius OP_1.

---

* The later argument assumes that $A \neq 0$. Since there are at most $n-1$ values of z_1 which make $A = 0$ (§ 77), such values may be avoided by taking some new point for P_1 on the circle of radius OP_1.

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